Combinatorics 1 – Problem Sheet 1

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Exercise 1

A *composition of* $n \in \mathbb{N}$ is a tuple of positive integers $(p_1, \dots, p_r)$ summing to n (unlike partitions, the summands do not have to be in decreasing order). Find a formula for the number of compositions of size n, and determine the corresponding generating function.

Exercise 2

Knowing the geometric series expansion

$$\frac{1}{1-z} = \sum_{n \geq 0} z^n,$$

find a closed form for the coefficients of the series

$$\frac{1}{(1-z)^2} \qquad \text{and} \qquad \frac{1}{(1-z)^3}.$$

What about $(1-z)^{-k}$?

Exercise 3

Recall that a *partition of* $n \in \mathbb{N}$ is a weakly decreasing sequence $p_1 \geq p_2 \geq \cdots \geq p_r$ of positive integers summing to n. For example, there are 5 partitions of the number 4:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Find a closed form expression for the number of partitions of n that use only the numbers $\{1, 2, 3\}$. Try to find as simple an answer as possible. *Hint:* Build the generating function, then try a partial fraction decomposition (possibly with complex numbers!). You can use a computer algebra system to compute partial fraction decompositions (for instance, in Maple use the command CONVERT(F,PARFRAC);)

Exercise 4

In class we saw how to take annihilating polynomials m_α and m_β of two algebraic numbers α and β and compute an annihilating polynomial $m_{\alpha+\beta}$ of $\alpha + \beta$ using resultants. Show how to use resultants to compute annihilating polynomials of $\alpha - \beta, \alpha\beta$, and (when $\beta \neq 0$) α/β. Conclude that the set of algebraic numbers form a subfield of $\mathbb{C}$.

Exercise 5

Let A be a commutative ring and

$$P(z) = p_n z^n + \cdots + p_1 z + p_0$$
$$Q(z) = q_m z^m + \cdots + q_1 z + q_0$$

be polynomials of degree n and m in $A[z]$. The *Sylvester matrix* $S(P, Q)$ of P and Q is the $(m+n) \times (m+n)$ matrix obtained by repeating the vector $(p_n, \ldots, p_0)$ m times with each copy shifted once over, then repeating the vector $(q_m, \ldots, q_0)$ n times with each copy shifted once over. For instance, if $n = 4$ and $m = 3$ then

$$S(P,Q) = \begin{pmatrix} p_4 & p_3 & p_2 & p_1 & p_0 & 0 & 0 \\ 0 & p_4 & p_3 & p_2 & p_1 & p_0 & 0 \\ 0 & 0 & p_4 & p_3 & p_2 & p_1 & p_0 \\ q_3 & q_2 & q_1 & q_0 & 0 & 0 & 0 \\ 0 & q_3 & q_2 & q_1 & q_0 & 0 & 0 \\ 0 & 0 & q_3 & q_2 & q_1 & q_0 & 0 \\ 0 & 0 & 0 & q_3 & q_2 & q_1 & q_0 \end{pmatrix}.$$

The *resultant* $\mathrm{res}_z(P, Q)$ of P and Q is the determinant of $S(P, Q)$.

a) Show that $\mathrm{res}_z(P, Q)$ lies in A. (Very easy with this definition!)

b) Prove that if P and Q share a root α then $\mathrm{res}_z(P, Q) = 0$. *Hint:* By the definition of the determinant it is sufficient to prove there is a non-zero vector $\mathbf{x}$ such that $S(P, Q)\mathbf{x} = \mathbf{0}$.

c) Suppose that

$$P(z) = a(z - \alpha_1) \cdots (z - \alpha_n) \qquad \text{and} \qquad Q(z) = b(z - \beta_1) \cdots (z - \beta_m).$$

Prove that $\mathrm{res}_z(P, Q) = R$ where

$$R = a^m b^n \prod_{i,j} (\alpha_i - \beta_j).$$

Hint: Consider $\mathrm{res}_z(P, Q)$ and R to be multivariate polynomials in new variables α_i and β_j. Show that R divides the resultant and consider their degrees.

d) Conclude that the resultant lies in A and is zero if and only if P and Q share a root in some algebraic closure of A.

Exercise 6: Bonus puzzle

Consider the following game. I pick a polynomial with natural number coefficients which you know nothing about (not even its degree). You can give me an integer, and I will tell you the value of the polynomial when evaluated at that integer. Show that there is a constant C such that you can always determine my polynomial exactly by giving me at most C values to evaluate (you can pick which values to ask me depending on my response to previous queries). What is the smallest value of C possible? Does it depend on the polynomial?