MATH 581 Assignment 1

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Exercise 1: Algebraic Numbers and Resultants

In class we saw how to take annihilating polynomials m_α and m_β of two algebraic numbers α and β and compute an annihilating polynomial $m_{\alpha+\beta}$ of $\alpha + \beta$ using resultants. Show how to use resultants to compute annihilating polynomials of $\alpha - \beta, \alpha\beta$, and (when $\beta \neq 0$) α/β. Conclude that the set of algebraic numbers form a subfield of $\mathbb{C}$.

Exercise 2: Words Avoiding Runs

Fix integers $m, k \geq 1$. Prove that the generating function $F(z)$ counting words (finite sequences) on an alphabet $\mathcal{A} = \{a_1, \ldots, a_m\}$ with m letters that do not contain k consecutive occurrences of a_1 is

$$F(z) = \frac{1 - z^k}{1 - mz + (m-1)z^{k+1}}.$$

Hint: Try an induction.

Exercise 3: Restricted Partitions

Recall that a *partition of* $n \in \mathbb{N}$ is a weakly decreasing sequence $p_1 \geq p_2 \geq \cdots \geq p_r$ of positive integers summing to n. For example, there are 5 partitions of the number 4:

$$4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1.$$

Find a closed form expression for the number of partitions of n that use only the numbers $\{1, 2, 3\}$. Your answer should involve only n, rational numbers, and the third root of unity $\rho = e^{2\pi i/3}$. You may use a computer algebra system to compute a partial fraction decomposition if desired (for instance, in Maple use the command CONVERT(F,PARFRAC);)

Exercise 4: Dominant Singularities

Using the Cauchy Integral Formula, prove that an analytic function defined by a convergent power series $F(z) = \sum_{n \geq 0} f_n z^n$ in a neighbourhood of the origin with radius of convergence $R > 0$ has a singularity on the circle $|z| = R$.

Exercise 5: Even Values of Riemann Zeta

Let $F(z) = z/(e^z - 1)$. Using an argument similar to our treatment of $\tan(z)$ in class, prove

$$f_n := [z^n]F(z) = -\sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i k)^n}.$$

Recall the Riemann zeta function

$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s}.$$

Show that $f_n = 0$ for n odd but when $n = 2m$ is an even positive integer then

$$f_{2m} = (-1)^{m-1} 2^{1-2m} \pi^{-2m} \cdot \zeta(2m).$$

Prove that $\zeta(2m)$ is a rational multiple of π^{2m}, and describe an algorithm to compute it.

Exercise 6: Fibonacci Polynomials

Define the *Fibonacci polynomials* over a field K of characteristic 0 by the recurrence $F_0(x) = 1, F_1(x) = x$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$. How many operations $(+, -, \times, \div)$ in K does the naive algorithm use to determine all coefficients of $F_N(x)$? Find an algorithm which uses $O(M(N))$ operations to compute all coefficients of $F_N(x)$, where $M(N)$ is the cost of multiplying two polynomials of degree at most N.

Exercise 7: Natural Boundary

Prove that the function $F(z) = \sum_{n \geq 0} z^{2^n}$ cannot be analytically continued outside of the disk $|z| < 1$.

Exercise 8: False Proof

What is wrong with the "proof"

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = \left(\sqrt{-1}\right)^2 = -1?$$

Exercise 9: Unmarked bonus challenge

Find an algorithm that computes all coefficients of the Fibonacci polynomial $F_N(x)$ in $O(N)$ operations.

Exercise 10: Unmarked bonus challenge

Consider the following game. I pick a polynomial with natural number coefficients which you know nothing about (not even its degree). You can give me an integer, and I will tell you the value of the polynomial when evaluated at that integer. Show that there is a constant C such that you can always determine my polynomial exactly by giving me at most C values to evaluate (you can pick which values to ask me depending on my response to previous queries). What is the smallest value of C possible?