MATH 581 Assignment 2

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Exercise 1: Transcendence of Binomial Powers

Prove that for any natural number $\kappa \geq 2$,

$$F(z) = \sum_{n\geq 0} \binom{2n}{n}^{\kappa} z^n$$

is transcendental. You may use any of the analytic, algebraic, or asymptotic properties of algebraic functions discussed in class or in the manuscript. This was conjectured by Stanley and originally proven by Flajolet.

Exercise 2: Binary Strings

Solve the following problem, posed in the December 2011 Mathematical Monthly.

> **11610**. *Proposed by Richard P. Stanley, Massachussetts Institute of Technology, Cambridge, MA.* Let $f(n)$ be the number of binary words $a_1 \cdots a_n$ of length n that have the same number of pairs $a_i a_{i+1}$ equal to 00 as equal to 01. Show that
>
> $$\sum_{n=0}^{\infty} f(n) t^n = \frac{1}{2}\left(\frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}}\right).$$

Hint: Let $Z(a,b,c)$ and $O(a,b,c)$ be the generating functions for the number of binary words ending in a 0 or 1 (respectively) where a counts the length of the word, b counts the number of 00 patterns, and c counts the number of 01 patterns. Set up and solve a system of equations involving Z and O, then use an integral calculation to determine the desired generating function.

Exercise 3: Diagonals Over Finite Fields

Find an algebraic equation satisfied by

$$F(z) = \Delta\left(\frac{1}{(1-w-x)(1-y-z)}\right) = \sum_{n\geq 0} \binom{2n}{n}^2 z^n$$

over the finite field $\mathbb{F}_p$ of order p for any prime p. Note $F(z)$ is transcendental over $\mathbb{C}[[z]]$.

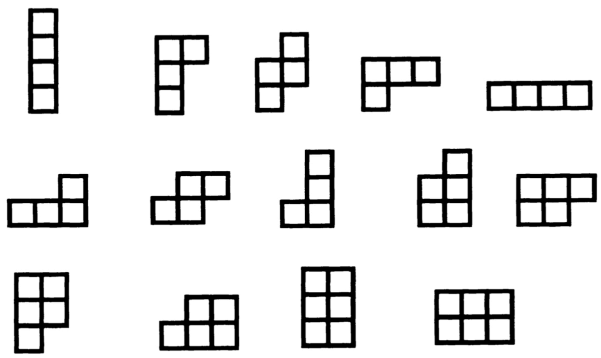
Hint: The 'Freshman's dream' states that for any polynomial $f(x)$ with integer coefficients, $f(x)^p = f(x^p)$ modulo p. Using this, show

$$\binom{2np+2r}{np+r} = \binom{2n}{n}\binom{2r}{r} \mod p.$$

1 Choice of Exercises

Do ***two*** *exercises in this section. When attempting an exercise you may assume the results of all previous parts, including those you do not solve yourself. If you solve more than two exercises, your best solutions will be taken into consideration.*

A *path pair* of length n is a pair of paths starting at the origin, consisting of n unit steps to the north or east, meeting again for the first time after n steps. The 14 pairs of length 5 are as follows.



Let $\mathcal{S}_n$ denote all pairs of paths of length n. An open problem several decades old is whether or not the elements of $\mathcal{S}_n$ tile a $2^{n-2} \times 2^{n-2}$ chessboard. We will show that the elements of $\mathcal{S}_n$ cover 4^{n-2} boxes, meaning this is plausible. To that end, let $\mathfrak{S}_n$ denote the class of lattice walks of length n in $\mathbb{Z}$ which start at the origin, take the steps $\mathcal{T} = \{(1,1), (1,-1), (1,0), (1,0)\}$ (there are two different horizontal steps), end at height 0, and touch the x-axis only at their beginning and end.

1. Find a bijection between $\mathcal{S}_n$ and $\mathfrak{S}_n$, such that $\boldsymbol{\rho} \in \mathcal{S}_n$ is paired to an element $\mathbf{w} \in \mathfrak{S}_n$ so that the area of $\boldsymbol{\rho}$ (number of squares covered) corresponds to area under the walk $\mathbf{w}$ (number of integer points under the walk and above or on the x-axis). See Figure 1.

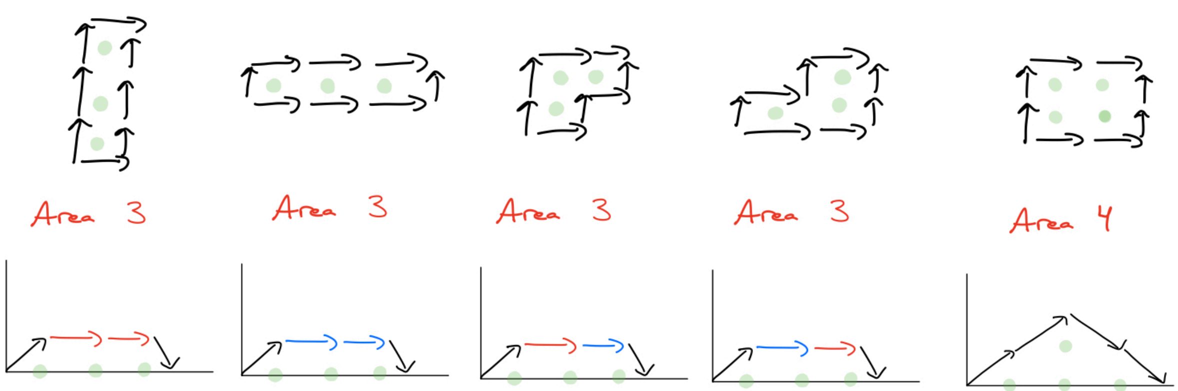


Figure 1: The walks and path pairs of length 4 in $\mathcal{S}_n$ and $\mathfrak{S}_n$, with areas marked.

2. Let G be the trivariate generating function

$$G(y,u,t) = \sum_{j,k,n\geq 0} g_{jkn} y^j u^k t^n,$$

where g_{jkn} denotes the number of walks in $\mathfrak{S}_n$ which end at $y = k$ and have area k. Prove that $G(y,u,t) = ut^2 F(y,u,ut)$, where $F(y,u,t)$ satisfies the kernel-like equation

$$F(y,u,t) = 1 + t\left(yu + 2 + \frac{1}{yu}\right) F(yu,u,t) - \frac{t}{yu} F(0,u,t).$$

Hint: G counts only walks which touch the x-axis in their first and last steps. F is the generating function of a less restricted class.

3. Using this kernel equation, prove

$$F(0,1,t) = \frac{2}{1 - 2t + \sqrt{1-4t}}$$

and

$$F(y,1,t) = \frac{2}{1 - 2t + \sqrt{1-4t} - 2ty}.$$

4. By differentiating the kernel equation and doing basic algebra, prove

$$F_u(0,1,t) = \frac{1 - 4t + 2t^2 + (2t-1)\sqrt{1-4t}}{2(1-4t)t^2}.$$

5. Using $F(0,1,t)$ and $F_u(0,1,t)$ show that the generating function for the number of boxes filled by all walks of length n is $t^2/(1-4t)$, proving the elements of $\mathcal{S}_n$ cover the right area for the conjecture to hold.

Unmarked bonus: Find a tiling of the chessboard with the elements of $\mathcal{S}_n$ for $n = 5$ (I would suggest printing out the last page and cutting out the pieces so you can move them around). Can you do $n = 6, 7, \ldots$?

Unmarked bonus: Prove that if a rectangle R can be tiled by smaller rectangles each having at least one side of integer length then R has at least one side of integer length. (All rectangles have parallel sides) There is a slick solution using (complex) integration!