Invitation to Analytic Combinatorics – Exercise Sheet 1

Exercise 1: Generic Recurrence Behaviour

Consider the recurrence relation

$$3f_{n+2} = 7f_{n+1} - 2f_n \qquad \text{for all } n \geq 0$$

with the initial conditions $f_0 = a$ and $f_1 = b$ for constants a and b. Find a closed form for f_n in terms of the constants a and b. What conditions on a and b make $f_n \to 0$ as $n \to \infty$ and what conditions on a and b make $|f_n| \to \infty$ as $n \to \infty$?

Exercise 2: Linear Diophantine Equations

Let $a_1, \dots, a_r$ be (not necessarily distinct) positive integers with greatest common divisor 1. Find dominant asymptotics for the number of solutions s_n to the equation

$$n = a_1x_1 + \cdots + a_tx_t \text{ with } x_k \in \mathbb{N} \text{ for all } 1 \leq k \leq r.$$

Exercise 3: Transcendence of Binomial Powers

Prove that for any natural number $\kappa \geq 2$,

$$F(z) = \sum_{n\geq 0} \binom{2n}{n}^{\kappa} z^n$$

is transcendental. You may use any of the analytic, algebraic, or asymptotic properties of algebraic functions discussed. *Hint: You might want to use the facts that* $\Gamma(1/2) = \sqrt{\pi}$ *and* $\Gamma(z+1) = z\Gamma(z)$*, and that* π *is transcendental.*

Exercise 4: Computing Even Zeta Values

Let $F(z) = z/(e^z - 1)$. Following our treatment of $\tan(z)$ in lecture, prove

$$f_n = [z^n]F(z) = -\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi i k)^n}.$$

You may use without proof the fact that $e^z - 1$ is uniformly bounded away from 0 when z is in any square in the complex plane with corners at $\pi(2k+1)(\pm 1 \pm i)$ with $k \in \mathbb{N}$.

Recall the Riemann zeta function

$$\zeta(s) = \sum_{k\geq 1} \frac{1}{k^s}.$$

Show that $f_n = 0$ for n odd but when $n = 2m$ is an even positive integer then

$$f_{2m} = (-1)^{m-1}2^{1-2m}\pi^{-2m} \cdot \zeta(2m).$$

Prove that $\zeta(2m)$ is a rational multiple of π^{2m}, and describe an algorithm to compute it.

Exercise 5: Singularities on Domain of Convergence Boundary

Use the Cauchy Integral Formula and the Maximum Modulus Bound to prove that if $F(z) = \sum_{n\geq 0} f_n z^n$ has finite radius of convergence $R > 0$ then $F(z)$ admits a singularity with modulus R.