

Invitation to Analytic Combinatorics – Exercise Sheet 2

Exercise 1: G-Functions

- a) Prove that if $F(z) = \Delta \left(\frac{G(z)}{H(z)} \right)$ is a rational diagonal with $G, H \in \mathbb{Q}[z]$ then there exist non-zero integers $a, b \in \mathbb{Z}$ such that $aF(bz)$ has integer coefficients.
- b) Prove Eisenstein's Theorem: if $F(z)$ is an algebraic series with rational coefficients then there exist non-zero integers $a, b \in \mathbb{Z}$ such that $aF(bz)$ has integer coefficients.

Exercise 2: A Bivariate Generating Function

For all $m, n \in \mathbb{N}$ let $c_{m,n}$ denote the number of solutions to

$$|z_1| + \cdots + |z_m| \leq n, \quad z_1, \dots, z_m \in \mathbb{Z}.$$

Find the generating function $C(x, y) = \sum_{m,n \geq 0} c_{m,n} x^m y^n$ and prove that $c_{m,n} = c_{n,m}$.

Exercise 3: Binary Strings

Solve the following problem, posed in the December 2011 Mathematical Monthly.

11610. Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA. Let $f(n)$ be the number of binary words $a_1 \cdots a_n$ of length n that have the same number of pairs $a_i a_{i+1}$ equal to 00 as equal to 01. Show that

$$\sum_{n=0}^{\infty} f(n)t^n = \frac{1}{2} \left(\frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}} \right).$$

Hint: Let $F(a, b, c)$ and $G(a, b, c)$ be the generating functions for the number of binary words ending in a 0 or 1 (respectively) where a counts the length of the word, b counts the number of 00 patterns, and c counts the number of 01 patterns. Set up a system of equations then take a “diagonal” in some of the variables: you can (and should) use a computer algebra system to help with calculations.

Exercise 4: Diagonals Over Finite Fields

Find an algebraic equation satisfied by

$$F(z) = \Delta \left(\frac{1}{(1-w-x)(1-y-z)} \right) = \sum_{n \geq 0} \binom{2n}{n} z^n$$

over the finite field \mathbb{F}_p of order p for any prime p . Note $F(z)$ is transcendental over $\mathbb{C}[[z]]$.

Hint: The ‘Freshman’s dream’ states that for any polynomial $f(x)$ with integer coefficients, $f(x)^p = f(x^p)$ modulo p . Using this, show that for $0 \leq r < p-1$,

$$\binom{2np+2r}{np+r} = \binom{2n}{n} \binom{2r}{r} \pmod{p}.$$