# Invitation to Analytic Combinatorics – Exercise Sheet 3

Let $(q_n)$ denote the number of walks on $\mathbb{Z}^2$ beginning at the origin, taking $n$ steps in

$$\{(1,0),(-1,0),(0,1),(0,-1)\}$$

and staying in the non-negative quadrant $\mathbb{N}^2$. Let $(a_{i,j,n})$ count the number of these walks which start at the origin and end at $(i,j) \in \mathbb{N}$.

**Exercise 1: A Functional Equation**

Prove that

$$A(x,y,t) = 1 + t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)A(x,y,t) - \frac{t}{x}A(0,y,t) - \frac{t}{y}A(x,0,t). \tag{1}$$

Hint: Decompose a walk of length $n$ as a walk of length $n-1$ plus an additional step.

**Exercise 2: An Orbit Sum Equation**

Equation (1) implies

$$\left(1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right)xyA(x,y,t) = xy - tyA(0,y,t) - txA(x,0,t). \tag{2}$$

Use this to prove

$$xyA(x,y,t) - \frac{x}{y}A\left(x,\frac{1}{y},t\right) + \frac{1}{xy}A\left(\frac{1}{x},\frac{1}{y},t\right) - \frac{y}{x}A\left(\frac{1}{x},y,t\right) = \frac{xy - \frac{x}{y} + \frac{1}{xy} - \frac{y}{x}}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}.$$

*Hint:* What is invariant under the substitutions $x \mapsto \frac{1}{x}$ and $y \mapsto \frac{1}{y}$.

**Exercise 3: The Positive Part Operator**

Let $\mathbb{Q}\left[x,\frac{1}{x},y,\frac{1}{y}\right][[t]]$ denote the ring of power series in $t$ whose coefficients are Laurent polynomials in $x$ and $y$ (each coefficient of $t$ contains a finite number of monomials, with potentially negative integer powers). Define the operator $[x^{\geq 0}y^{\geq 0}] : \mathbb{Q}\left[x,\frac{1}{x},y,\frac{1}{y}\right][[t]] \to \mathbb{Q}[x,y][[t]]$ which takes an element

$$P(x,y,t) = \sum_{n\geq 0}\left(\sum_{i,j\in\mathbb{Z}} p_{i,j,n}x^iy^jt^n\right) \in \mathbb{Q}\left[x,\frac{1}{x},y,\frac{1}{y}\right][[t]]$$

and returns the terms with non-negative exponents,

$$[x^{\geq 0}y^{\geq 0}]P(x,y,t) := \sum_{n\geq 0}\left(\sum_{i,j\geq 0} p_{i,j,n}x^iy^jt^n\right).$$

Prove that

$$A(x,y,t) = [x^{\geq 0}y^{\geq 0}]\frac{(1-x^2)(1-y^2)}{x^2y^2\left(1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right)}.$$

**Exercise 4: A Diagonal Expression**

Given

$$P(x,y,t) = \sum_{n\geq 0}\left(\sum_{(i,j)\in\mathcal{A}_k} p_{i,j,n}x^iy^jt^n\right) \in \mathbb{Q}\left[x,\frac{1}{x},y,\frac{1}{y}\right][[t]],$$

show that

$$[x^{\geq 0}y^{\geq 0}]P(x,y,t)\Big|_{x=1,y=1} = \Delta\left[\frac{P\left(\frac{1}{x},\frac{1}{y},xyt\right)}{(1-x)(1-y)}\right].$$

Infer that

$$Q(t) = \Delta\left[\frac{(1+x)(1+y)}{1-t\left(x+\frac{1}{x}+y+\frac{1}{y}\right)}\right]. \tag{3}$$

**Exercise 5: Further Models**

What changes in this argument if you replace the step set $\{(1,0),(-1,0),(0,1),(0,-1)\}$ with one of

$$\{(1,1),(1,-1),(-1,1),(-1,-1)\}$$

$$\{(0,1),(0,-1),(1,1),(1,-1),(-1,1),(-1,-1)\}$$

$$\{(0,1),(0,-1),(1,0),(-1,0),(1,1),(1,-1),(-1,1),(-1,-1)\}$$