Exercise Sheet: An Invitation to Analytic Combinatorics in Several Variables

S. Melczer

See https://melczer.ca/ALEA22 for computer algebra notebooks to help with the computations for these exercises, further references, and talk slides.

Question 1: Delannoy Numbers

The *Delannoy number* $d_{a,b}$ is the number of paths from the origin $(0,0)$ to the point (a,b) using only the steps $\mathsf{N} = (0,1)$, $\mathsf{E} = (1,0)$, and $\mathsf{NE} = (1,1)$. A history of these numbers, and a survey of some applications, is given in [1].

(a) Prove the recurrence

$$d_{a,b} = \begin{cases} 1 & : \text{ if } a = 0 \text{ or } b = 0 \\ d_{a-1,b} + d_{a,b-1} + d_{a-1,b-1} & : \text{ otherwise} \end{cases}$$

Conclude that

$$D(x,y) = \sum_{a,b \geq 0} d_{a,b} x^a y^b = \frac{1}{1-x-y-xy}.$$

(b) The *central Delannoy numbers* defined by the main diagonal sequence $(d_{n,n})$ appear when enumerating certain posets, tilings of the Aztec diagmond, and alignments of DNA sequences, among other applications [1]. Use the Main Theorem of Smooth ACSV to find asymptotics of these numbers as the $(1,1)$-diagonal of $D(x,y)$. What are the critical points in the $(1,1)$ direction? Which are minimal?

(c) Use the Main Theorem of Smooth ACSV to find asymptotics of the (r,s)-diagonal of $D(x,y)$ for any $r, s > 0$.

References

[1] Why Delannoy numbers? C. Banderier and S. Schwer. Journal of Statistical Planning and Inference, 135(1), 2005. https://arxiv.org/abs/math/0411128

Question 2: Apéry Asymptotics

Recall from lecture that a key step in Apéry's proof of the irrationality of $\zeta(3)$ is determining the exponential growth of the sequence that can be encoded as the main diagonal of

$$F(x, y, z, t) = \frac{1}{1 - t(1 + x)(1 + y)(1 + z)(1 + y + z + yz + xyz)}.$$

Use the Main Theorem of Smooth ACSV to find dominant asymptotics of this sequence.

References

- A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$ A. van der Poorten, 1978. https://link.springer.com/article/10.1007/BF03028234

Question 3: Pathological Directions

(a) Find asymptotics of the (r, s)-diagonal of $F(x, y) = \frac{1}{1-x-xy}$ for any $0 < s < r$.

(b) What are the critical points of $F(x, y) = \frac{1}{1-x-xy}$ in the (r, s) direction when $0 < r \leq s$? Which are minimal? Characterize the behaviour of the (r, s) diagonal when $0 < r \leq s$.

Question 4: A Composition Limit Theorem

An *integer composition* of size $n \in \mathbb{N}$ is an ordered tuple of positive integers (of any length) that sum to n. A composition can be viewed as an integer partition where the order of the summands matters. Let $c_{k,n}$ denote the number of compositions of size n that contain k ones. For instance, the compositions of size three are

$$3 = 2 + 1 = 1 + 2 = 1 + 1 + 1$$

so $c_{1,3} = 2$ (because both $2 + 1$ and $1 + 2$ both contain a single one) while $c_{3,3} = c_{0,3} = 1$ and $c_{2,3} = 0$.

(a − optional) If you know the symbolic method, species theory, or similar enumerative constructions, prove that

$$C(u, x) = \sum_{n,k \geq 0} c_{k,n} u^k x^n = \frac{1 - x}{1 - 2x - (u - 1)x(1 - x)}.$$

(b) Prove that the distribution for the number of ones in a composition of size n satisfies a local central limit theorem as $n \to \infty$. More precisely, find a constant $t > 0$ and normal density $\nu_n(s)$ such that

$$\sup_{s \in \mathbb{Z}} |t^n c_{s,n} - \nu_n(s)| \to 0$$

as $n \to \infty$.

Question 5: Quadrant Walks and ACSV

Define the sequences

$$q_n = \# \text{ walks in } \mathbb{N}^2 \text{ on } n \text{ steps in } \{\mathsf{N},\mathsf{S},\mathsf{E},\mathsf{W}\} = \{(\pm 1,0),(0,\pm 1)\} \text{ starting at } \mathbf{0}$$

$$w_{i,j,n} = \# \text{ walks in } \mathbb{N}^2 \text{ on } n \text{ steps in } \{\mathsf{N},\mathsf{S},\mathsf{E},\mathsf{W}\} \text{ starting at } \mathbf{0} \textbf{ and ending at } (i,j) \in \mathbb{N}^2$$

We use the refined sequence $w_{i,j,n}$ tracking the length and endpoint of a walk to find asymptotics of q_n.

(a) Show that the generating functions

$$Q(t) = \sum_{n\geq 0} q_n t^n \qquad \text{and} \qquad W(x,y,t) = \sum_{i,j,n\geq 0} w_{i,j,n} x^i y^j t^n$$

satisfy $Q(t) = W(1,1,t)$. Justify the functional equation

$$W(x,y,t) = 1 + t\left(x + \frac{1}{x} + y + \frac{1}{y}\right) W(x,y,t) - \frac{t}{x} W(0,y,t) - \frac{t}{y} W(x,0,t). \tag{1}$$

by decomposing a walk of length n as a walk of length $n-1$ plus an additional step.
Hint: How do you keep track of the restriction that a walk must stay in $\mathbb{N}^2$?

(b) Equation (1) implies

$$\left(1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right) xyW(x,y,t) = xy - tyW(0,y,t) - txW(x,0,t).$$

Use this to prove

$$xyW(x,y,t) - \frac{x}{y} W\left(x, \frac{1}{y}, t\right) + \frac{1}{xy} W\left(\frac{1}{x}, \frac{1}{y}, t\right) - \frac{y}{x} W\left(\frac{1}{x}, y, t\right) = \frac{xy - \frac{x}{y} + \frac{1}{xy} - \frac{y}{x}}{1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)}. \tag{2}$$

Remark: We can make the substitutions $x \mapsto \frac{1}{x}$ and $y \mapsto \frac{1}{y}$ as $W(x,y,t)$ is a power series in t whose coefficients are *polynomials* in x and y.

(c) Let $\mathcal{R}$ denote the ring of power series in t whose coefficients are *finite* sums of monomials in x and y (with potentially negative integer powers) and let $[x^{\geq 0} y^{\geq 0}]$ denote the operator that takes a series in $\mathcal{R}$ and returns the terms where all exponents of x and y are non-negative. For instance,

$$[x^{\geq 0} y^{\geq 0}]\left(2 + \left(1 + x + xy^{-1}\right)t + \left(y - x^{-2}y^3\right)t^2 + \cdots\right) = 2 + (1+x)t + yt^2 + \cdots .$$

Use Equation (2) to prove

$$W(x,y,t) = [x^{\geq 0} y^{\geq 0}] \, \frac{(1-x^2)(1-y^2)}{x^2y^2\left(1 - t\left(x + \frac{1}{x} + y + \frac{1}{y}\right)\right)}.$$

(d) Prove that for any $P(x,y,t) \in \mathcal{R}$ one has

$$[x^{\geq 0} y^{\geq 0}] P(x,y,t) \Big|_{x=1,y=1} = \Delta \left[\frac{P\left(\frac{1}{x}, \frac{1}{y}, xyt\right)}{(1-x)(1-y)} \right].$$

Infer that

$$Q(t) = W(1,1,t) = \Delta \left[\frac{(1+x)(1+y)}{1 - txy\left(x + \frac{1}{x} + y + \frac{1}{y}\right)} \right]. \tag{3}$$

(e) Use Equation (3) to find the dominant asymptotic behaviour of q_n by applying the main asymptotic theorem of smooth ACSV.

Remark: The asymptotic behaviour of q_n is not compatible with $Q(t)$ being an algebraic function. Thus (knowing a bit of analytic combinatorics) this computation proves $Q(t)$ is transcendental.

References

- Chapters 4 and 6 in An Invitation to Analytic Combinatorics: From One to Several Variables, S. Melczer, Springer International Publishing, 2021. `https://melczer.ca/textbook`
- Walks with small steps in the quarter plane, M. Bousquet-Mélou and M. Mishna, Contemporary Mathematics: Algorithmic probability and combinatorics, 2010. `https://arxiv.org/abs/0810.4387`