AN INVITATION TO ANALYTIC COMBINATORICS IN SEVERAL VARIABLES – LECTURE 1

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L'entrée du port de Marseille by Claude-Joseph Vernet (1754)

Topic 1
Analytic Combinatorics

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Enumerative Combinatorics

Start with a sequence

$$(f_n) = f_0, f_1, f_2, \ldots$$

The terms of the sequence could

- **count** objects in a combinatorial class
- capture the **probability** that an event occurs
- track the **runtime** of an algorithm

Goal: capture something **interesting** about the sequence

Exact Formulas

A THEOREM ON TREES.

By Prof. CAYLEY.

THE number of trees which can be formed with $n+1$ given knots $\alpha, \beta, \gamma, \ldots$ is $=(n+1)^{n-1}$; for instance $n=3$, the number of trees with the 4 given knots $\alpha, \beta, \gamma, \delta$ is $4^2=16$

A. Cayley. A Theorem on Trees. Quart. J. Pure Appl. Math. Vol 23, 376–378, 1889.

It's **unreasonable** to expect this to always occur — not all combinatorial sequences have *simple* formulas, and even if they do they can be hard to prove!

Algorithms

```
M = Matrix(ZZ,2,2,[1,1,1,0])

def bin_pow(n):
    if n == 1:      return M
    elif n%2 == 0: return bin_pow(n/2)^2
    else:           return bin_pow((n-1)/2)^2*M

def fib(n):         return add(bin_pow(n)[1])

timeit('fib(10^6)') # 208,987 digits long!
```

```
25 loops, best of 3: 10.1 ms per loop
```

Gathering data can be useful for studying sequences, and conjecturing formulas, but doesn't fully *capture behaviour.*

Asymptotics

Instead of exact enumeration, focus on **large-scale behaviour** by **approximating** f_n for large n.

$$\#\text{ partitions of } n \;\sim\; \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

$$\begin{array}{c}\text{average quicksort cost}\\ \text{on permutation of size } n\end{array} \;\sim\; 2n\log n$$

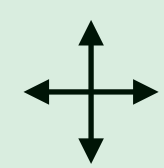
$$\#\text{ unlabelled graphs on } n \text{ nodes} \;\sim\; \frac{2^{\binom{2n}{n}}}{n!}$$

Generating Functions

The **generating function (GF)** of f_n is

$$F(z) = \sum_{n \geq 0} f_n z^n$$

Algebraic / differential / functional equations for F form a **data structure** for f_n

$f_n =$ # walks starting at $\mathbf{0}$ on n steps in 

$$F(z) = \frac{1}{1-4z}$$

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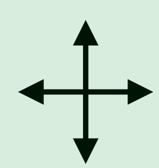
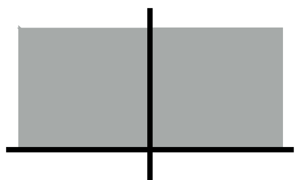
$f_n =$ # walks starting at $\mathbf{0}$ on  staying in halfspace

$$F(z) = \frac{4z + \sqrt{1-4z} - 1}{2z(1-4z)}$$



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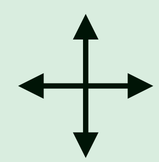
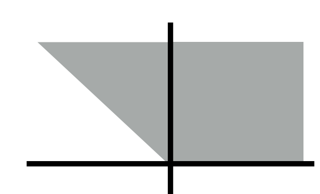
$f_n = $ # walks starting at $\mathbf{0}$ on  staying in $3\pi/4$ wedge

$$27z^7(4z-1)^2F(z)^8 + \cdots + (16z^2-12z+1)F(z) - 1 = 0$$



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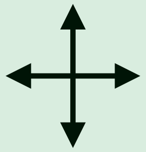
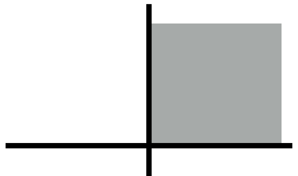
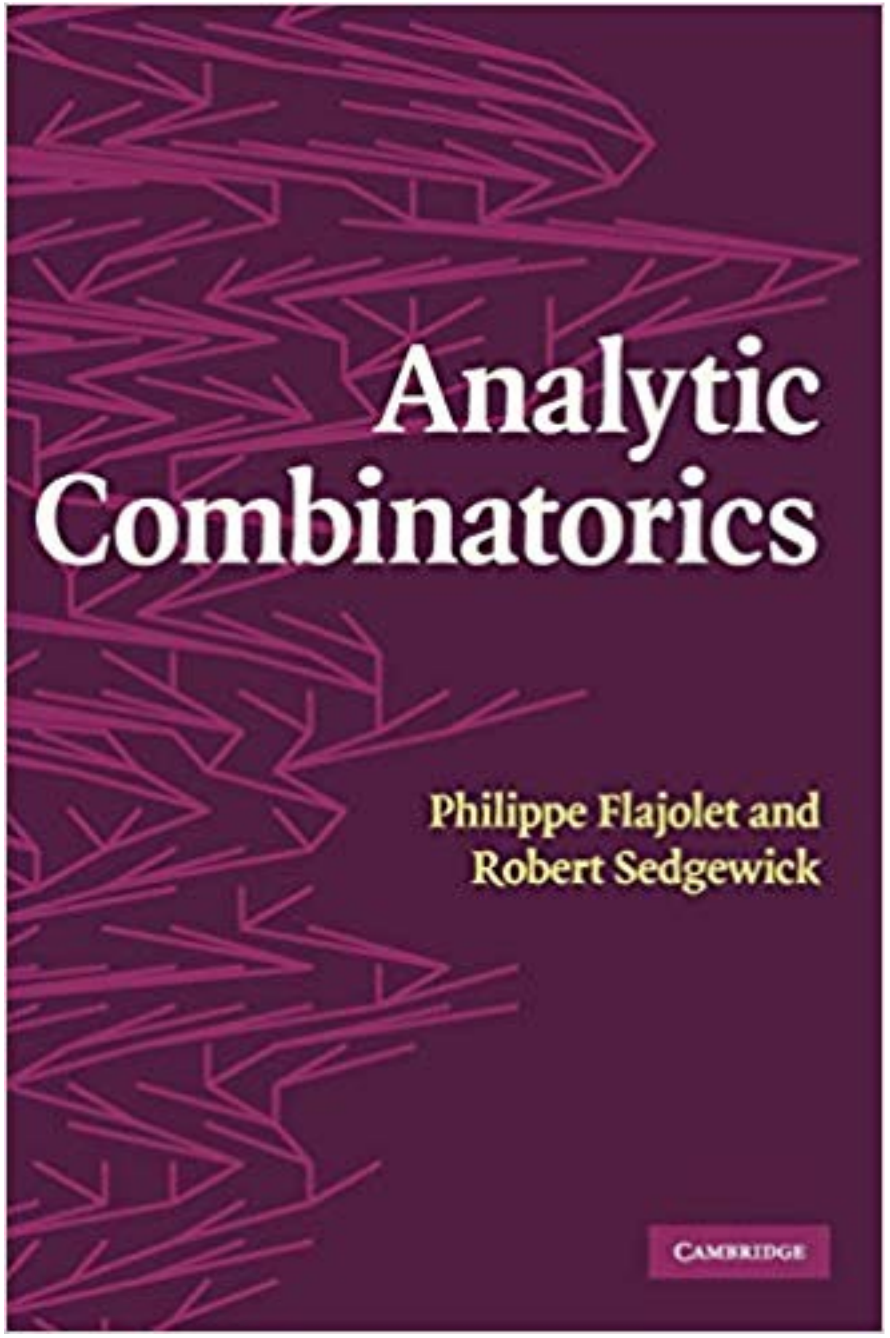
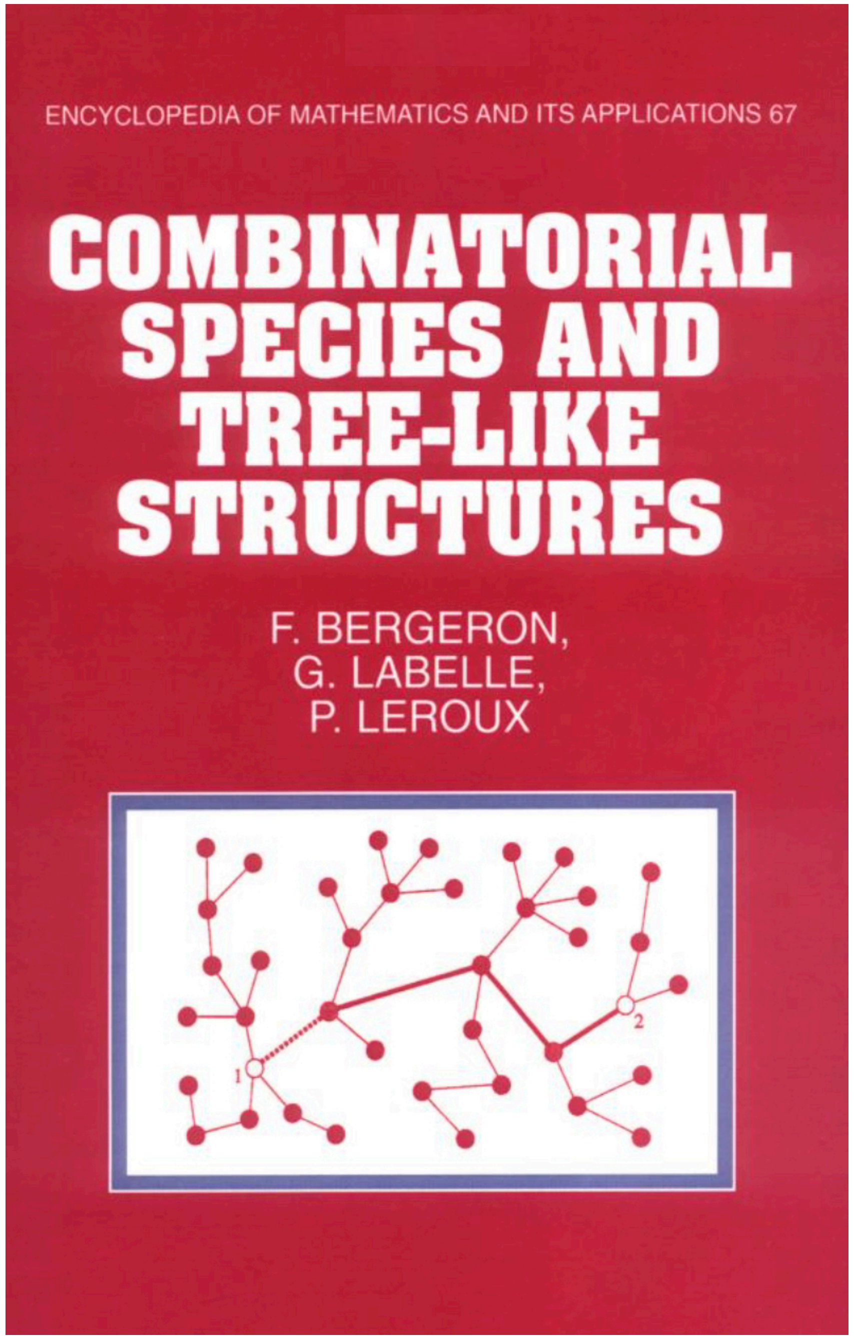
$f_n =$ # walks starting at **0** on  staying in quadrant

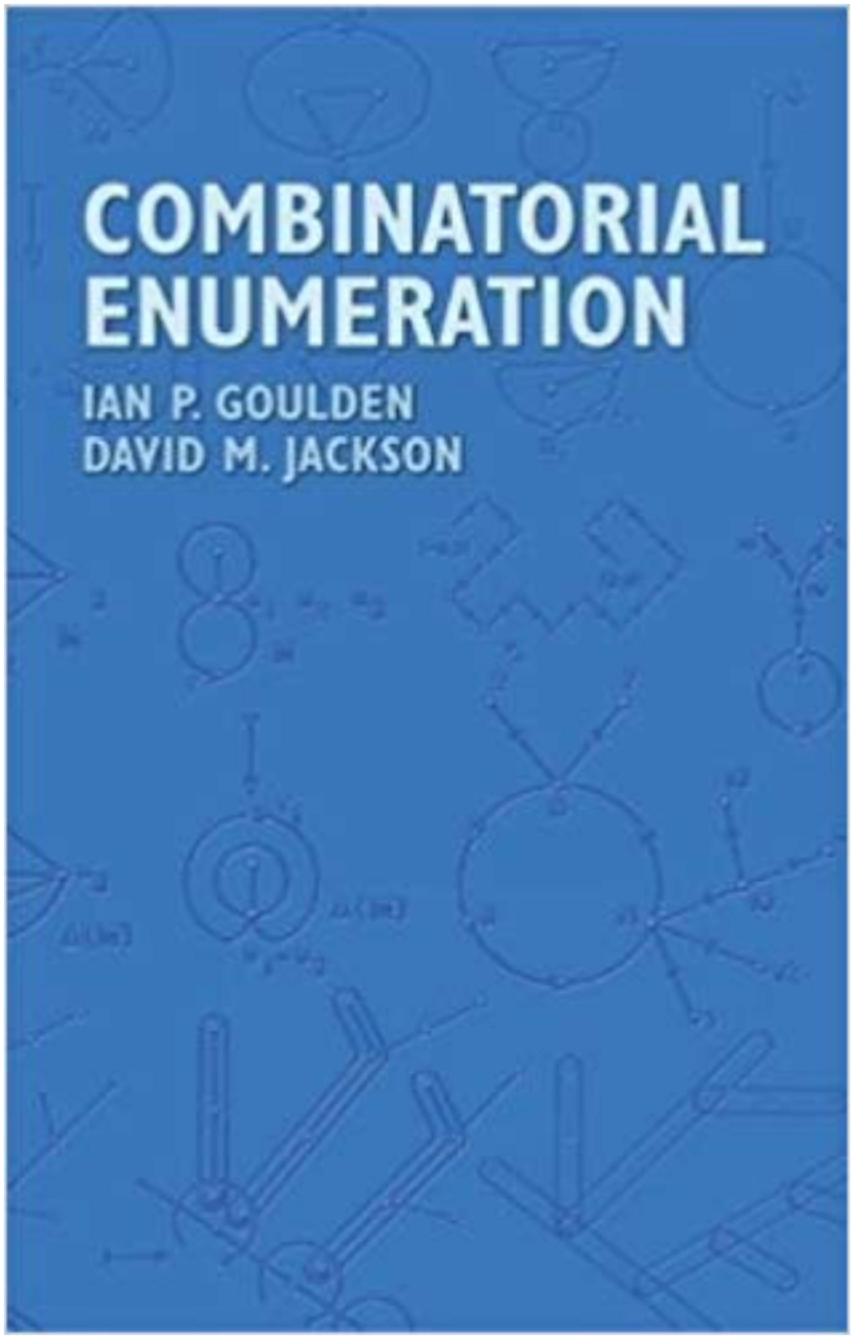
$$z^2(4z-1)(4z+1)F'''(z) + 2z(4z+1)(16z-3)F''(z) + 2(112z^2+14z-3)F'(z) + 4(16z+3)F(z) = 0$$



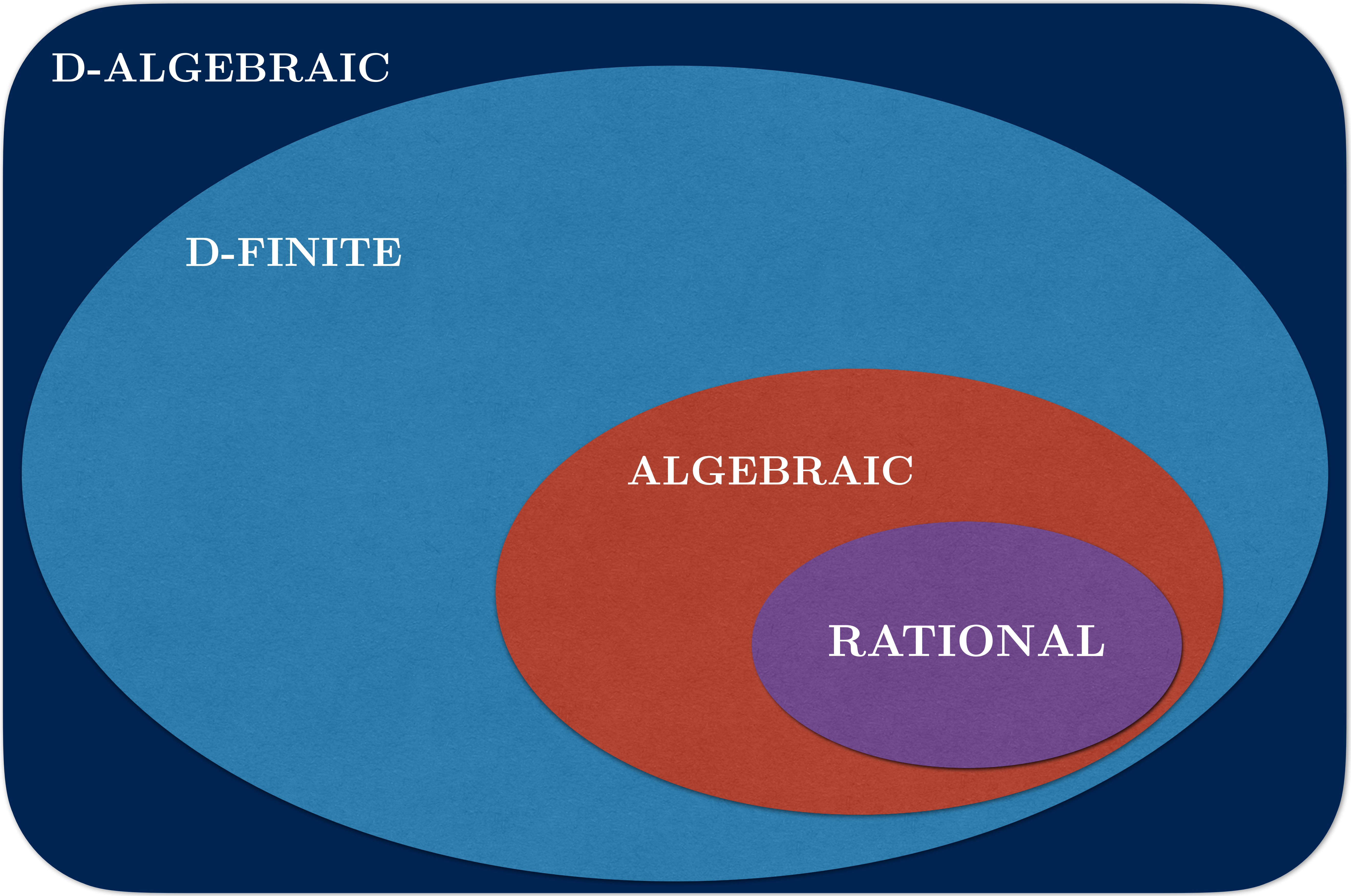
Combinatorial definitions often **automatically** translate to GF equations







Generating Function Classes



Analytic Combinatorics

We assume our GF is **analytic** at the origin

$$\sum_{n \geq 0} 2^n z^n \qquad \sum_{n \geq 0} \frac{z^n}{n}$$

$$\sum_{n \geq 0} n! \, z^n$$

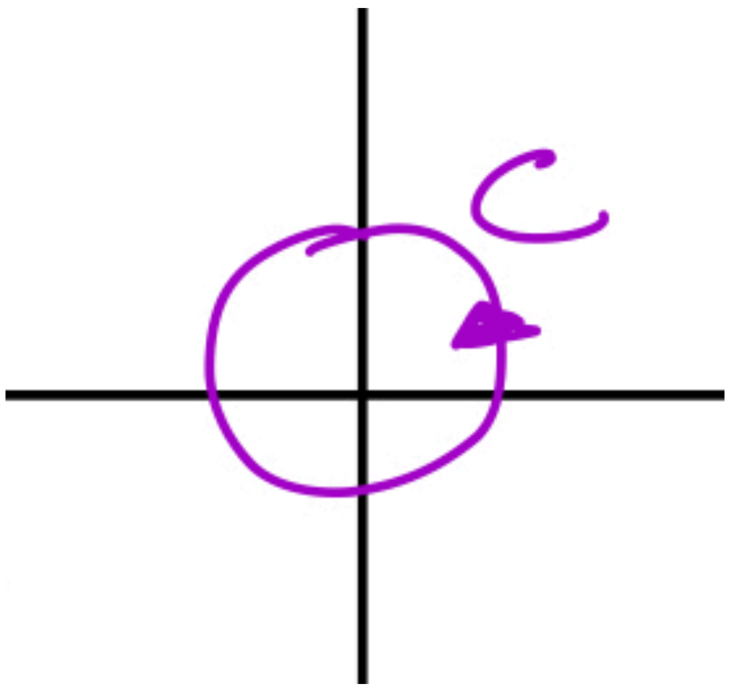
Analytic combinatorics derives asymptotics of sequences from the behaviour of their GFs

Fact 1: Cauchy Integral Formula

If $F(z) = \sum_{n \geq 0} f_n z^n$ is analytic at the origin then

$$f_n = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{F(z)}{z^{n+1}} dz$$

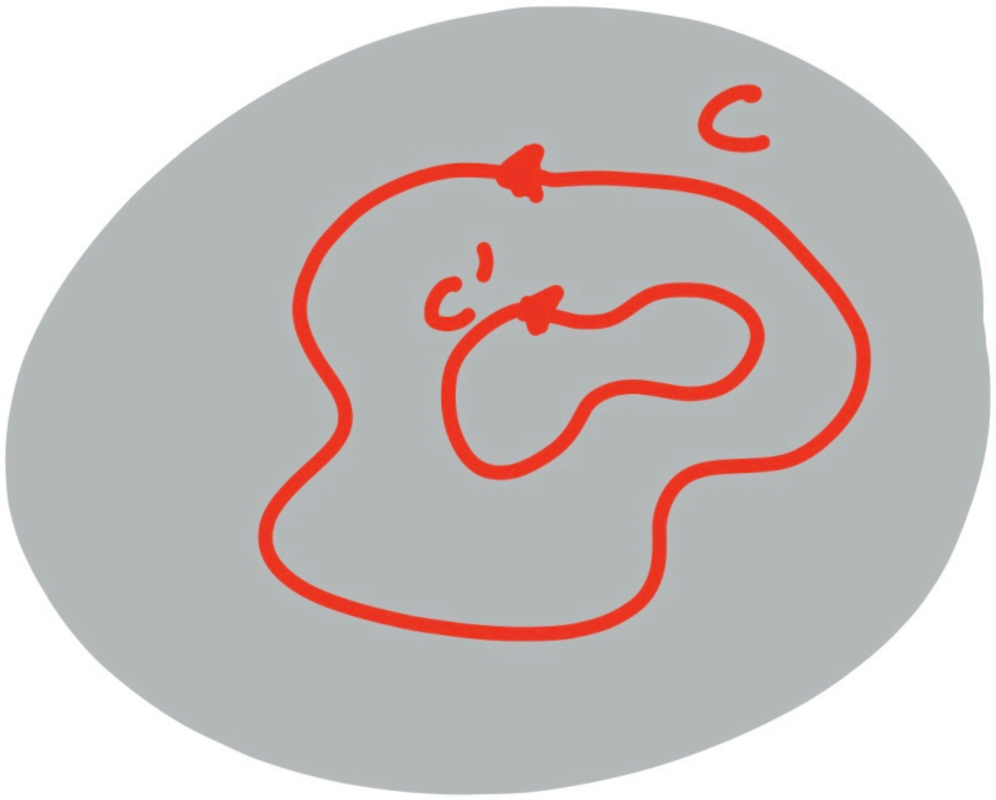
where $\mathcal{C}$ is a sufficiently small circle around the origin.



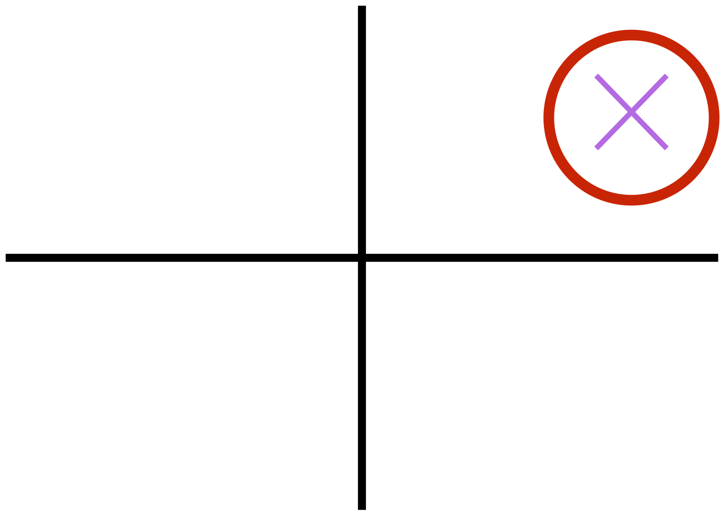
Fact 2: Deforming Curves of Integration

If $\mathcal{C}$ and $\mathcal{C}'$ are *simple closed curves* and $\mathcal{C}$ can be deformed to $\mathcal{C}'$ in an open set where $f(z)$ is analytic then

$$\int_{\mathcal{C}} f(z)dz = \int_{\mathcal{C}'} f(z)dz$$



Fact 3: Residues



Assume

- $P(z)$ and $Q(z)$ analytic at $z = \rho$
- $\mathcal{C}$ is any sufficiently small circle around $z = \rho$

Then

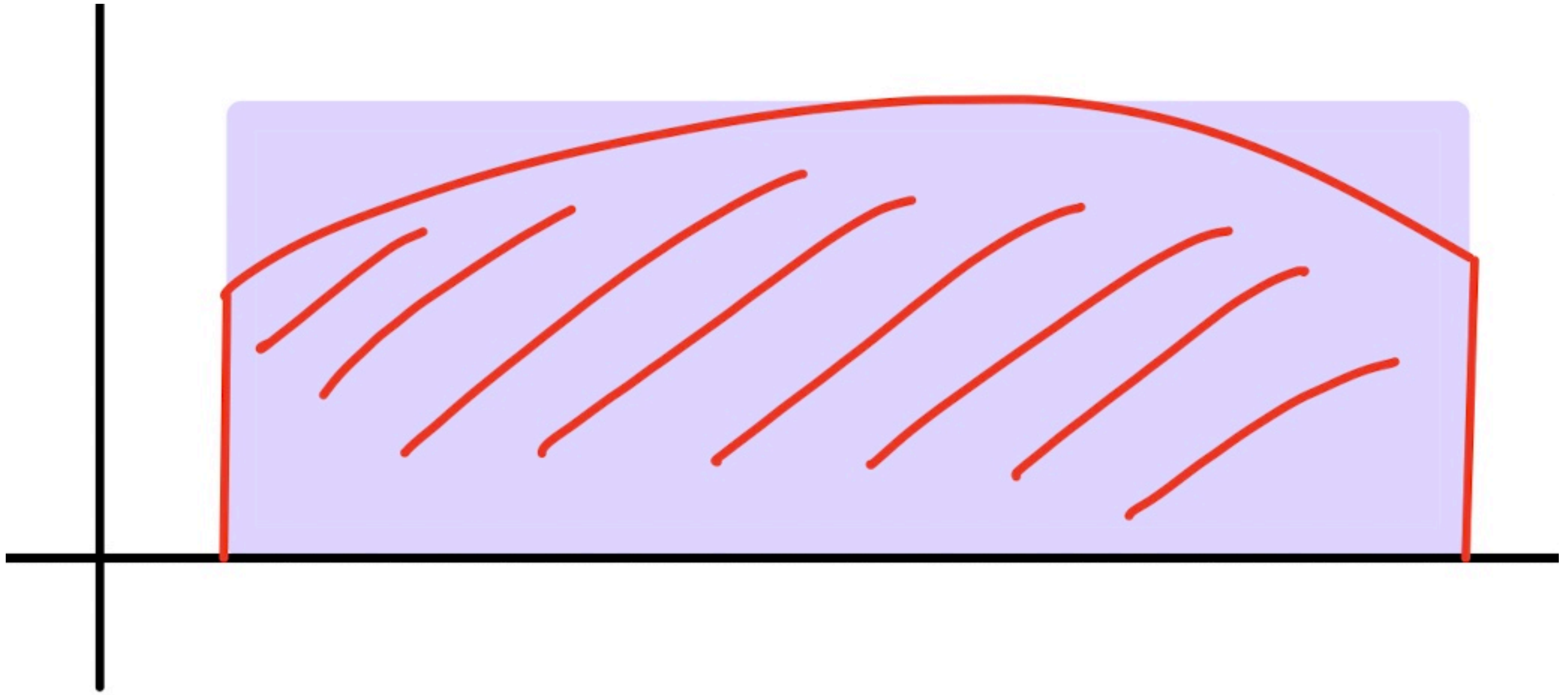
$$\int_{\mathcal{C}} \frac{P(z)}{Q(z)} dz = \operatorname*{Res}_{z=\rho} \left(\frac{P(z)}{Q(z)} \right)$$

where the **residue** is an explicit and computable expression involving the derivatives of P and Q at $z = \rho$

Fact 4: Max Modulus Bound

If $f(z)$ continuous on $\mathcal{C}$ then

$$\left|\int_{\mathcal{C}} f(z)dz\right| \leq \text{length}(\mathcal{C}) \times \max_{z\in\mathcal{C}} |f(z)|$$



Alternating Permutations

An **alternating permutation** is a permutation π of *odd length* such that $\pi_1 > \pi_2 < \pi_3 > \cdots$

The alternating permutations of length three: 213 and 312

$$A(z) = \sum_{k \geq 0} \frac{a_{2k+1}}{(2k+1)!} z^{2k+1}$$

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The alternating permutations of length three: 213 and 312

$$A(z) = \sum_{k \geq 0} \frac{a_{2k+1}}{(2k+1)!} z^{2k+1} = \tan z$$

ANALYSE MATHÉMATIQUE. — *Développements de* séc x *et de* tang x. Note de M. D. ANDRÉ, présentée par M. Hermite.

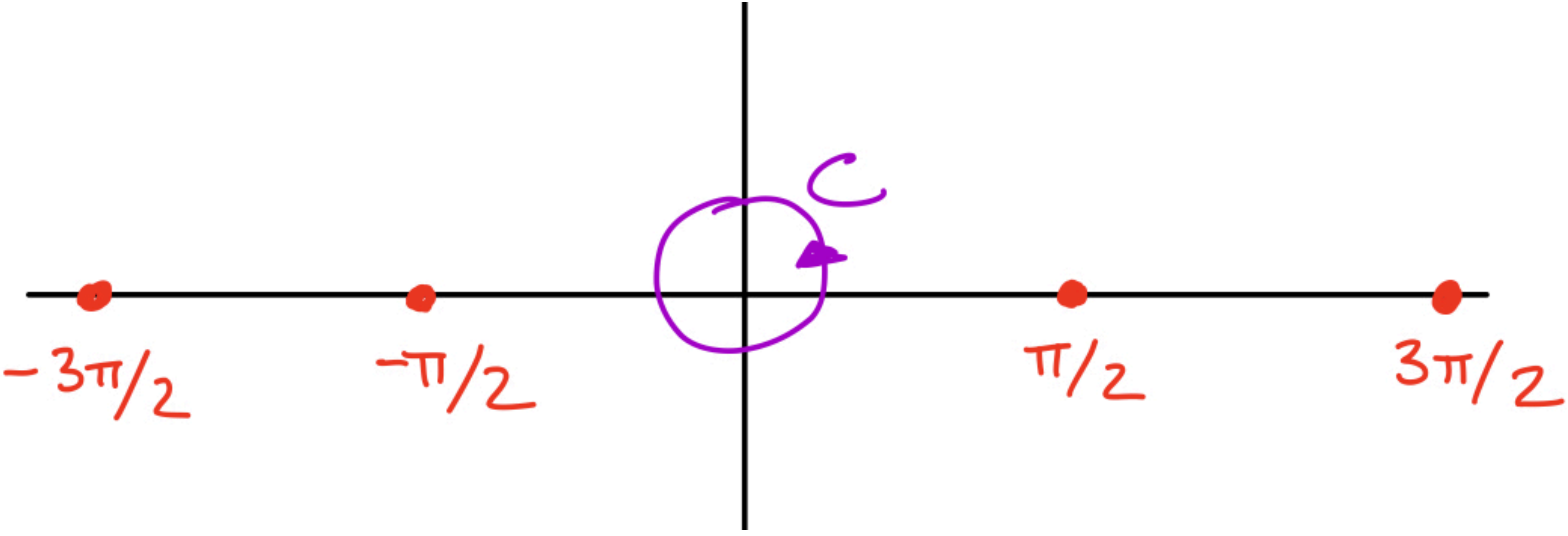
« On n'a point donné jusqu'à présent, du moins à ma connaissance, de développement, suivant les puissances de x, soit de tang x, soit de séc x, où les coefficients aient une définition simple, nette, indépendante de tout autre développement. L'objet de la présente Note est de combler cette lacune.

Comptes rendus de l'Académie des sciences, 1879

Asymptotics of Alternating Permutations

The **Cauchy integral formula** implies

$$\frac{a_n}{n!} = [z^n]\tan z = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\tan z}{z^{n+1}}$$



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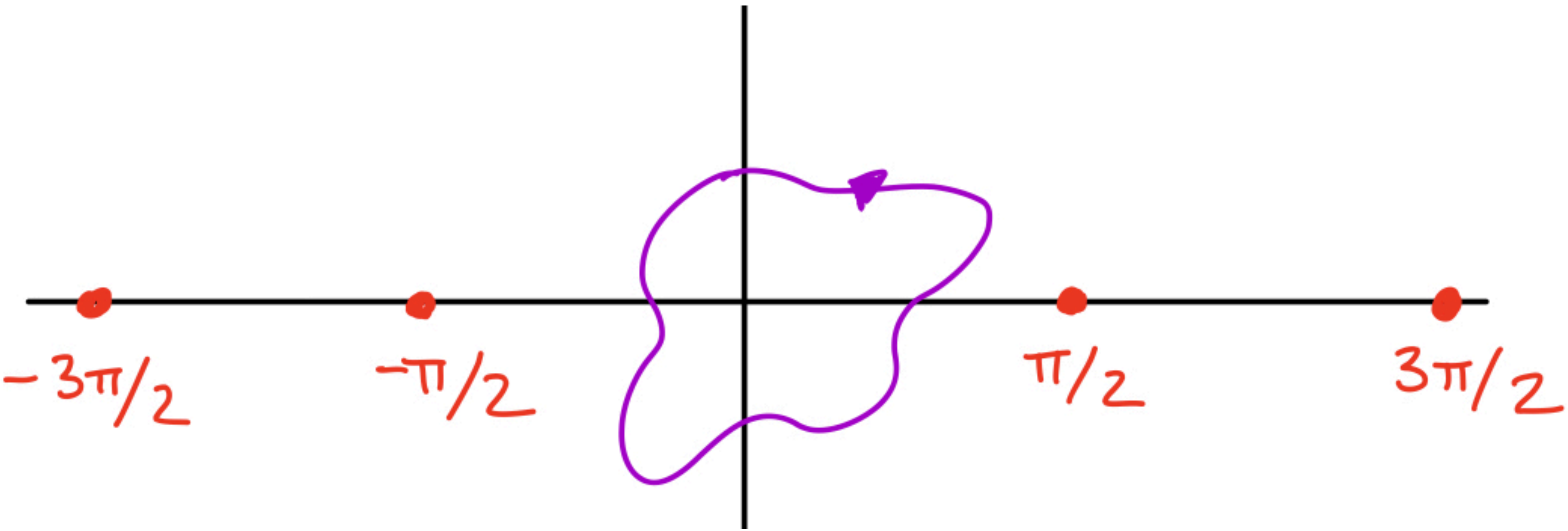

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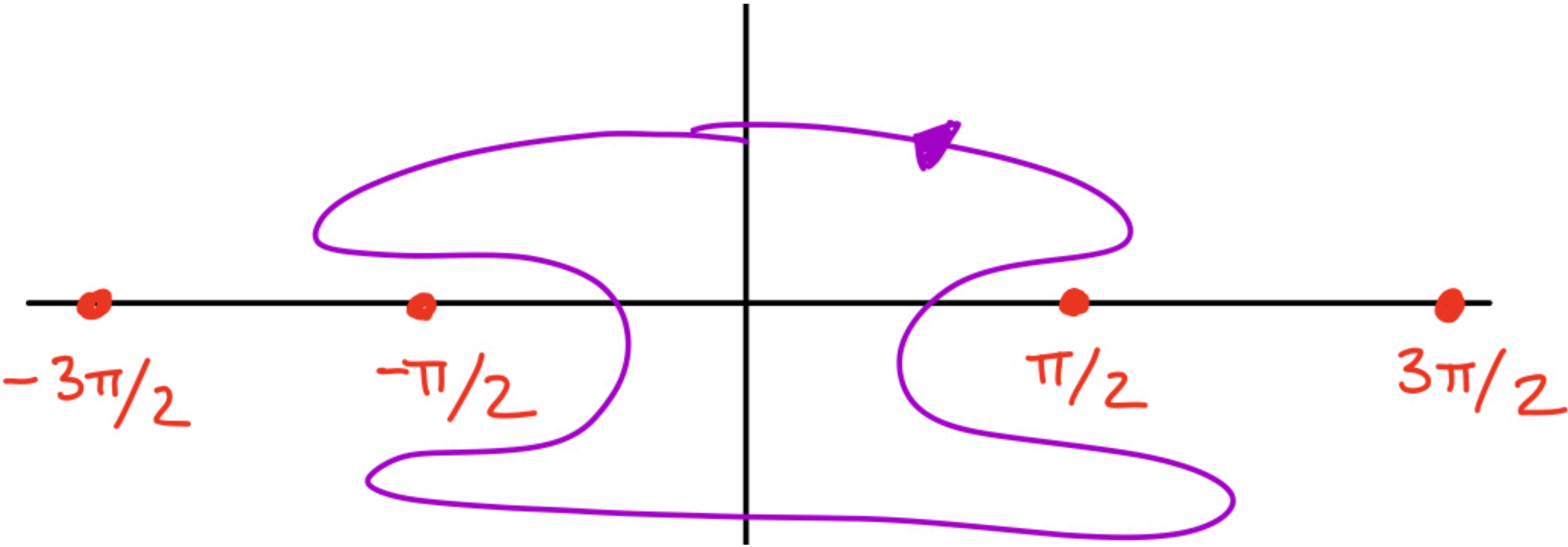

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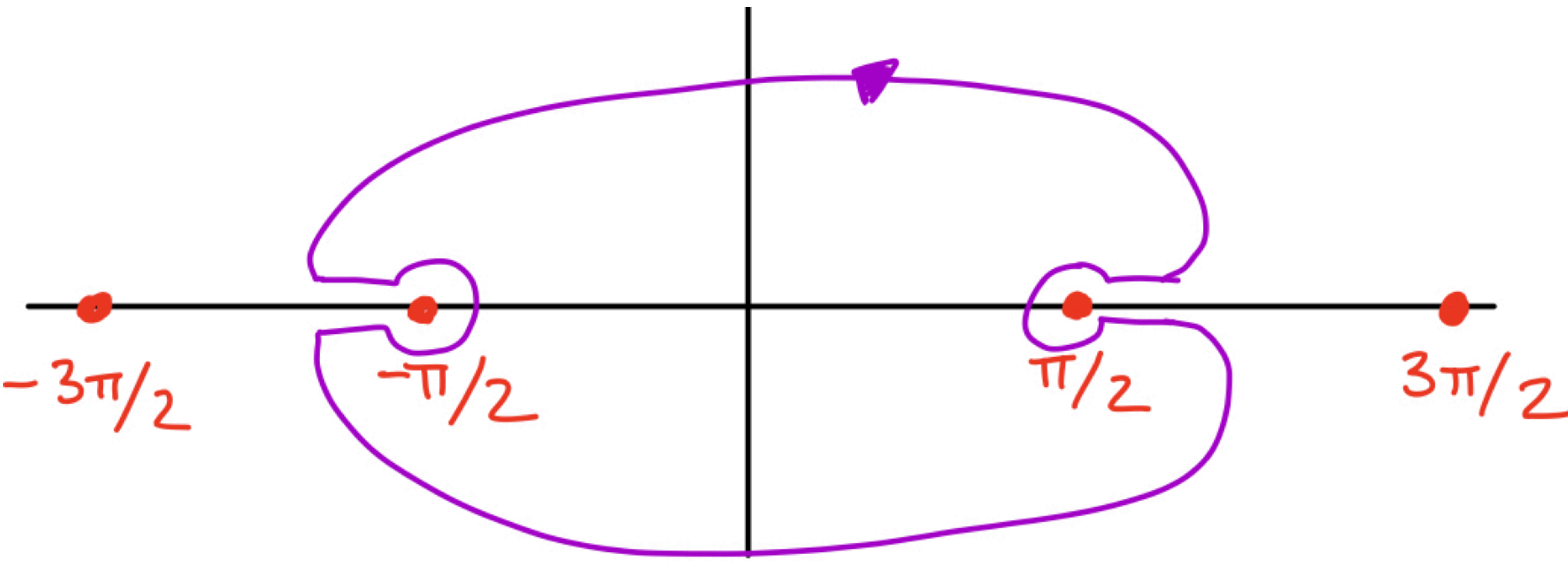

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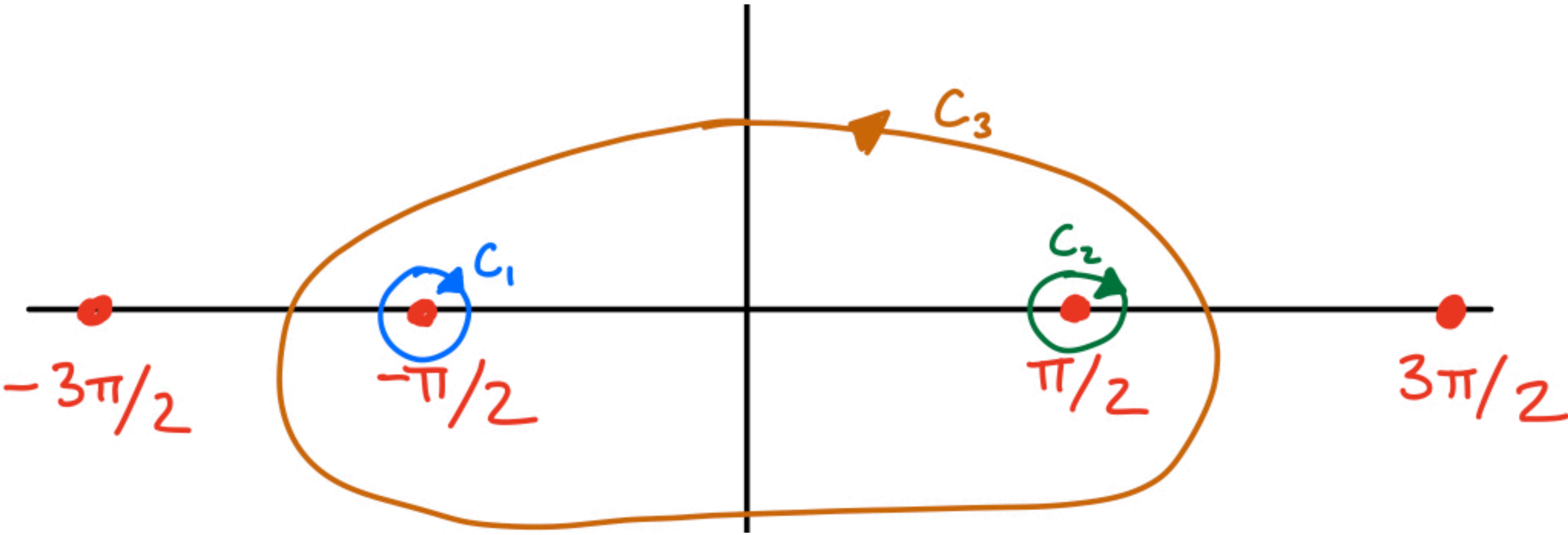
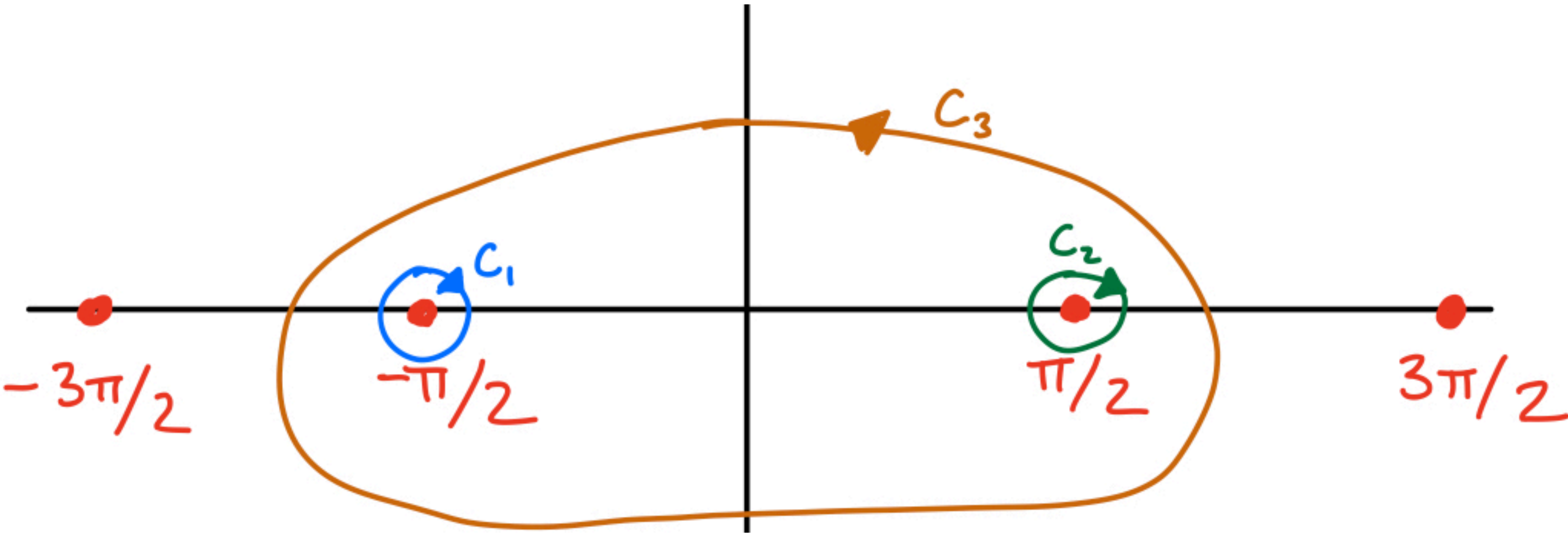


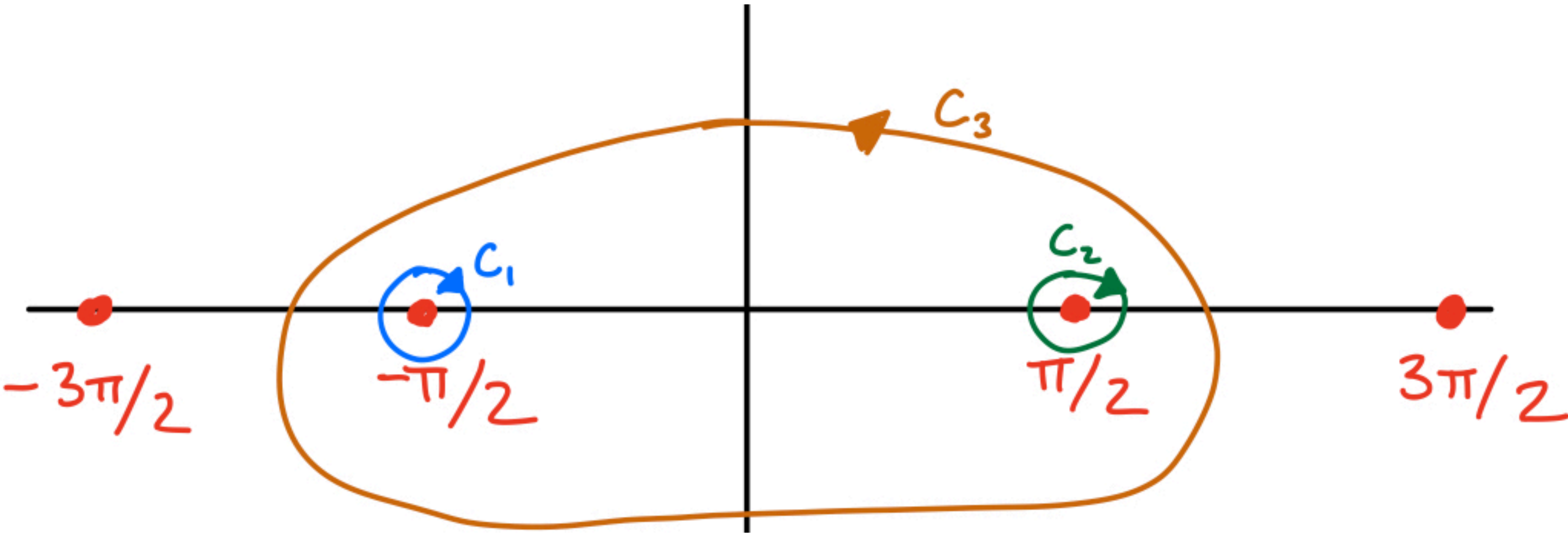
Asymptotics of Alternating Permutations

$$\frac{a_n}{n!} = \frac{1}{2\pi i}\int_{\mathcal{C}_1} \frac{\tan z}{z^{n+1}}dz + \frac{1}{2\pi i}\int_{\mathcal{C}_2} \frac{\tan z}{z^{n+1}}dz + \frac{1}{2\pi i}\int_{|z|=\pi} \frac{\tan z}{z^{n+1}}dz$$



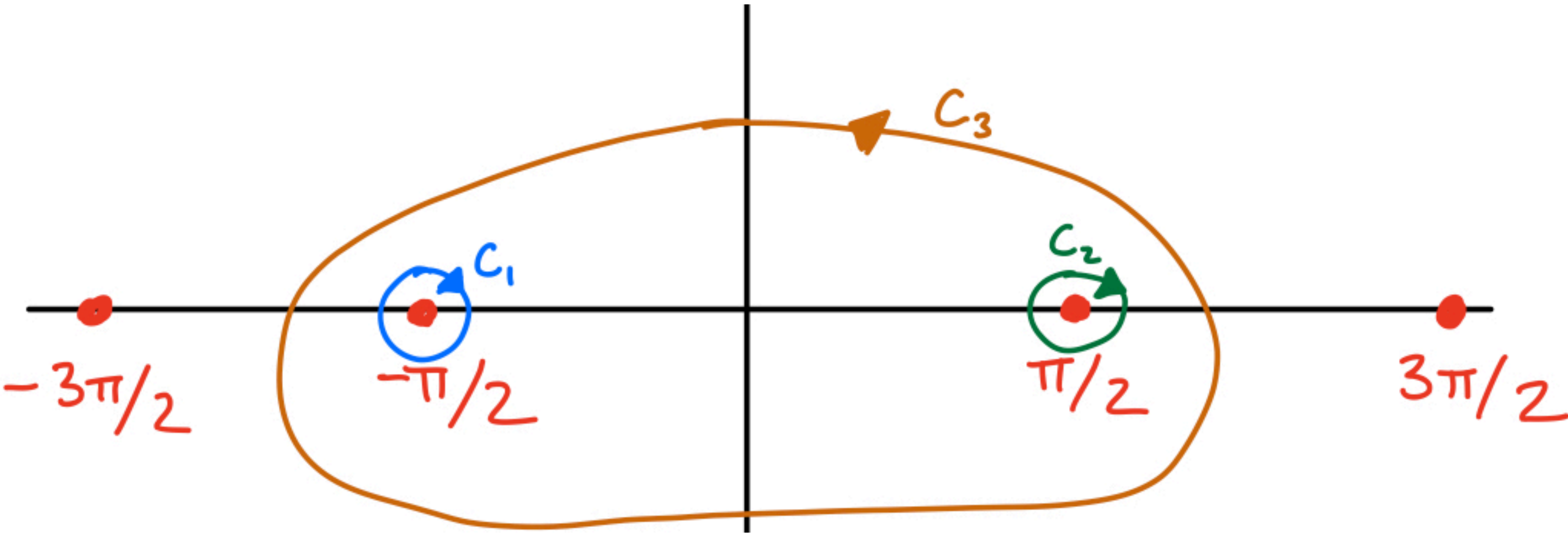
Asymptotics of Alternating Permutations

$$\frac{a_n}{n!} = \operatorname*{Res}_{z=-\pi/2}\left(\frac{\tan z}{z^{n+1}}\right) + \operatorname*{Res}_{z=\pi/2}\left(\frac{\tan z}{z^{n+1}}\right) + O\left(\left(\frac{1}{\pi}\right)^n\right)$$



Asymptotics of Alternating Permutations

$$\frac{a_n}{n!} = \left(\frac{-2}{\pi}\right)^{n+1} \operatorname*{Res}_{z=-\pi/2}\left(\frac{\sin z}{\cos z}\right) + \left(\frac{2}{\pi}\right)^{n+1} \operatorname*{Res}_{z=\pi/2}\left(\frac{\sin z}{\cos z}\right) + O\left(\left(\frac{1}{\pi}\right)^{n}\right)$$



Asymptotics of Alternating Permutations

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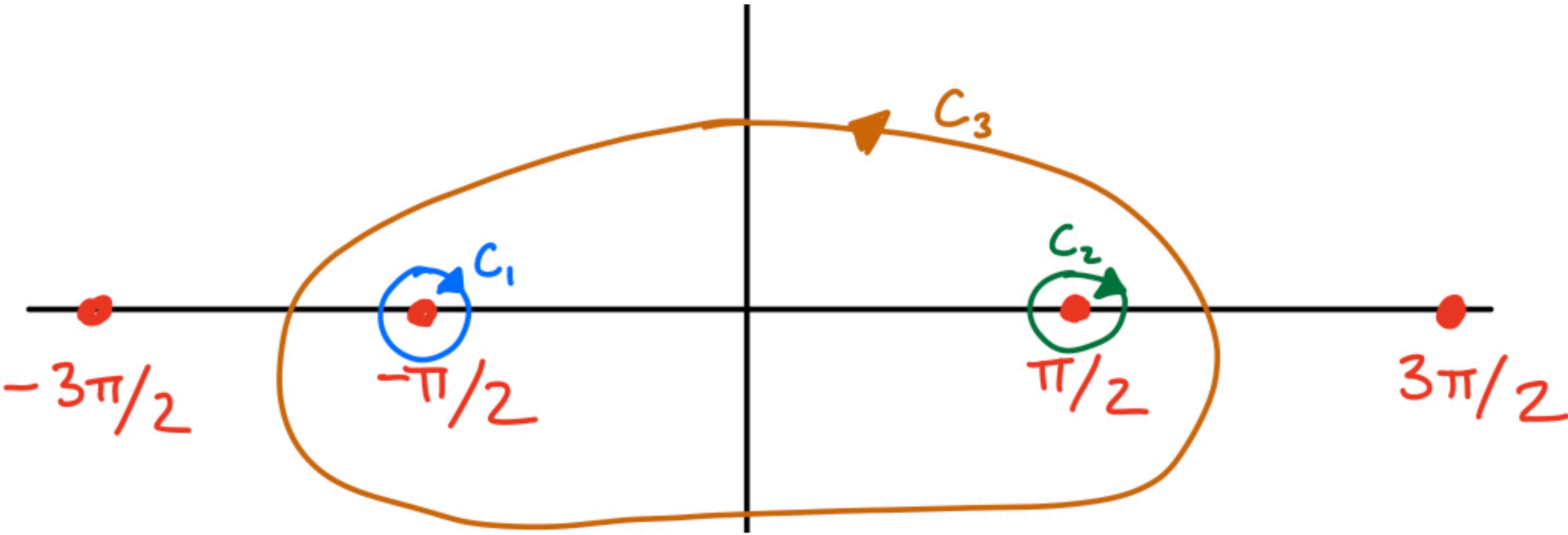
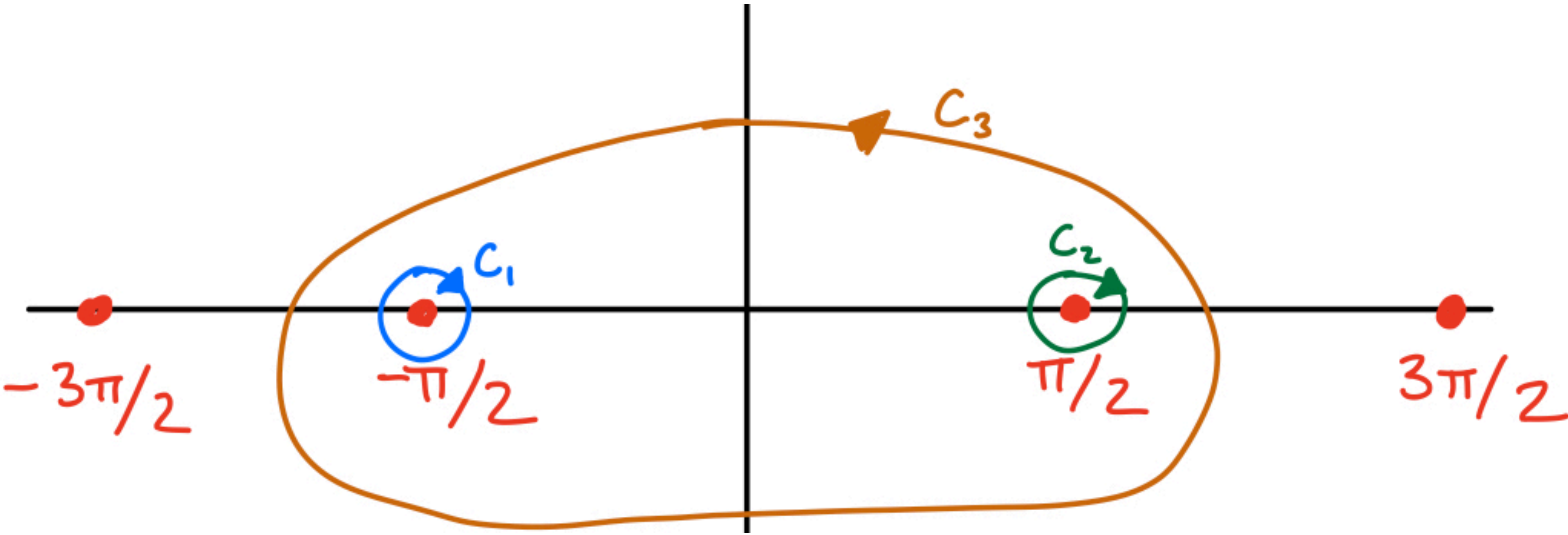


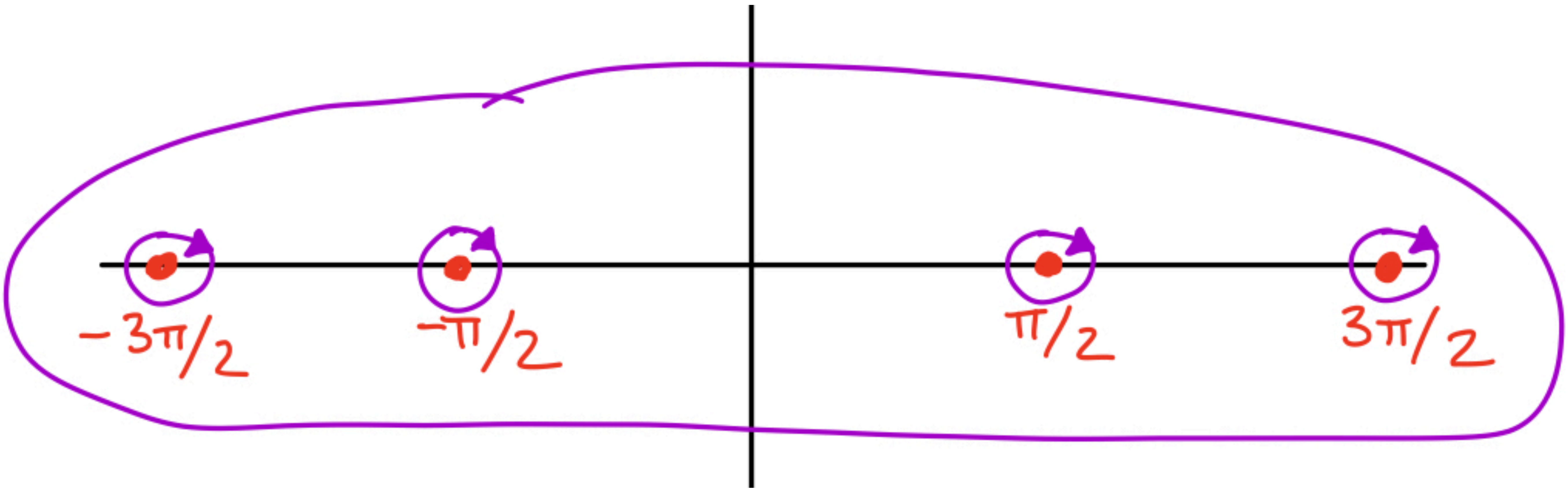
Asymptotics of Alternating Permutations

$$\frac{a_n}{n!} = 2\left(\frac{2}{\pi}\right)^{n+1} + O\left(\left(\frac{1}{\pi}\right)^n\right) \qquad (n \text{ odd})$$



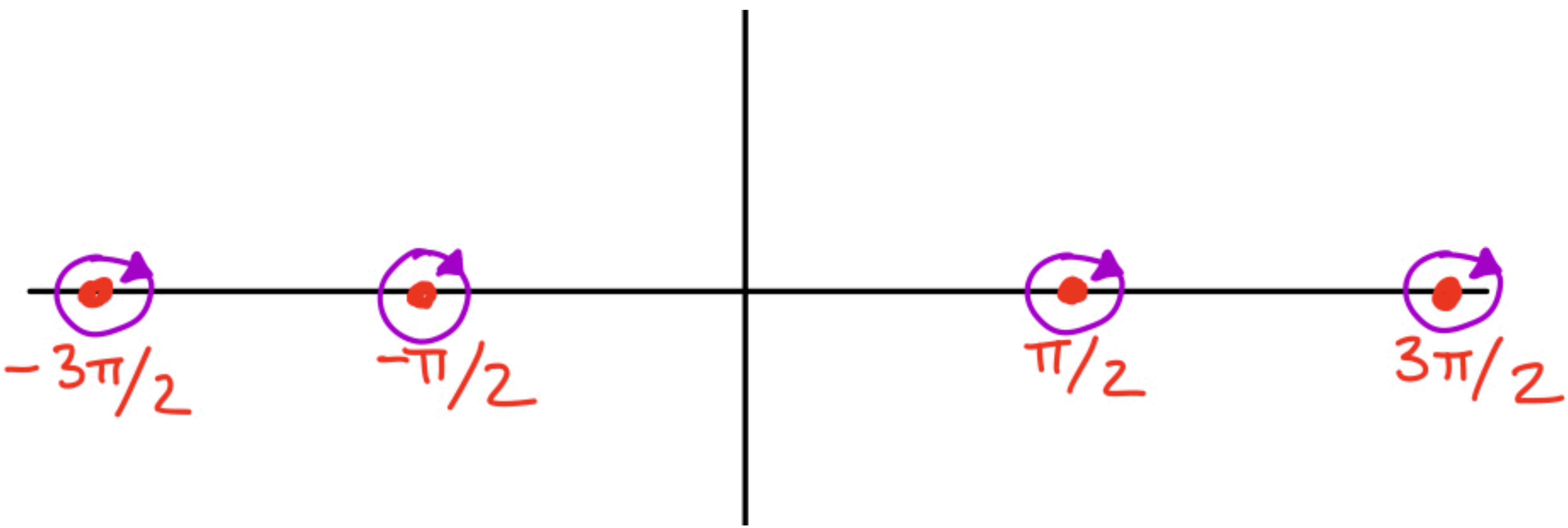
Asymptotics of Alternating Permutations

$$\frac{a_n}{n!} = 2\left(\frac{2}{\pi}\right)^{n+1} + 2\left(\frac{2}{3\pi}\right)^{n+1} + O\left(\left(\frac{2}{5\pi}\right)^{n}\right) \qquad (n \text{ odd})$$



Asymptotics of Alternating Permutations

$$\frac{a_n}{n!} = 2\left(\frac{2}{\pi}\right)^{n+1} \sum_{k\geq 0} \frac{1}{(2k+1)^{n+1}} \qquad (n \text{ odd})$$



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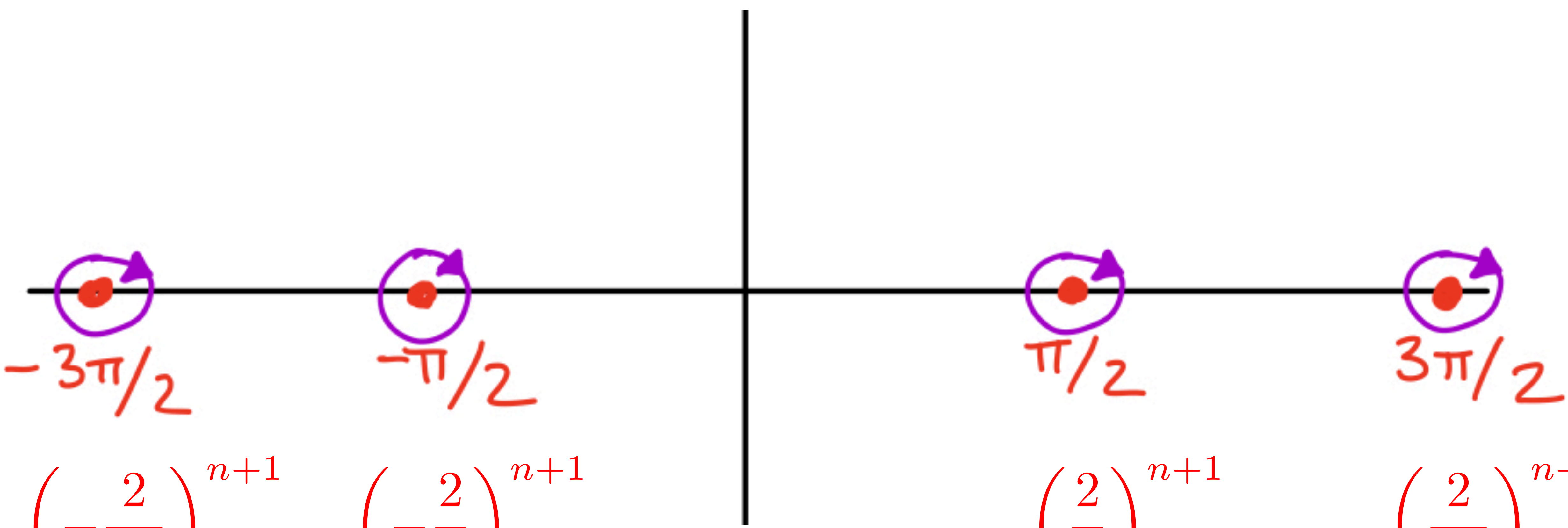

$$\left(-\frac{2}{3\pi}\right)^{n+1} \quad \left(-\frac{2}{\pi}\right)^{n+1} \quad \left(\frac{2}{\pi}\right)^{n+1} \quad \left(\frac{2}{3\pi}\right)^{n+1}$$

Main Takeaways

- Each **singularity** gives contribution
- Those singularities **closest to the origin** affect dominant asymptotics
- The contributions of each can be determined by a **local analysis** of the generating function

There are many **known formulas** for different types of singularities

$$F(z) \sim (1-z)^{\alpha} \left(\log \frac{1}{1-z} \right)^{\beta} \implies f_n \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} (\log n)^{\beta}$$

Topic 2
Diagonals and Smooth ACSV

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Diagonals

Start with a multivariate series

$$F(\mathbf{z}) = \sum_{(i_1,\ldots,i_d)\in\mathbb{N}^d} f_{i_1,\ldots,i_d} z_1^{i_1} \cdots z_d^{i_d} = \sum_{\mathbf{i}\in\mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

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The **r-diagonal** consists of the coefficients $(f_{n\mathbf{r}}) = f_{\mathbf{0}}, f_{\mathbf{r}}, f_{2\mathbf{r}}, \dots$
Note the coefficient $f_{n\mathbf{r}}$ is defined only if $n\mathbf{r} \in \mathbb{N}^d$

$(1,1)$ – **Diagonal (Main Diagonal)**

$$F(x,y) = \frac{1}{1-x-y}$$
$$= 1 + x + y + 2xy + x^2 + y^2 + 3x^2y + 3xy^2 + y^3 + 6x^2y^2 + \cdots$$

Diagonals

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$(2,1)$ – **Diagonal**

$$F(x,y) = \frac{1}{1-x-y}$$
$$= 1 + x + y + 2xy + x^2 + y^2 + 3x^2y + \cdots + 15x^4y^2 + \cdots$$

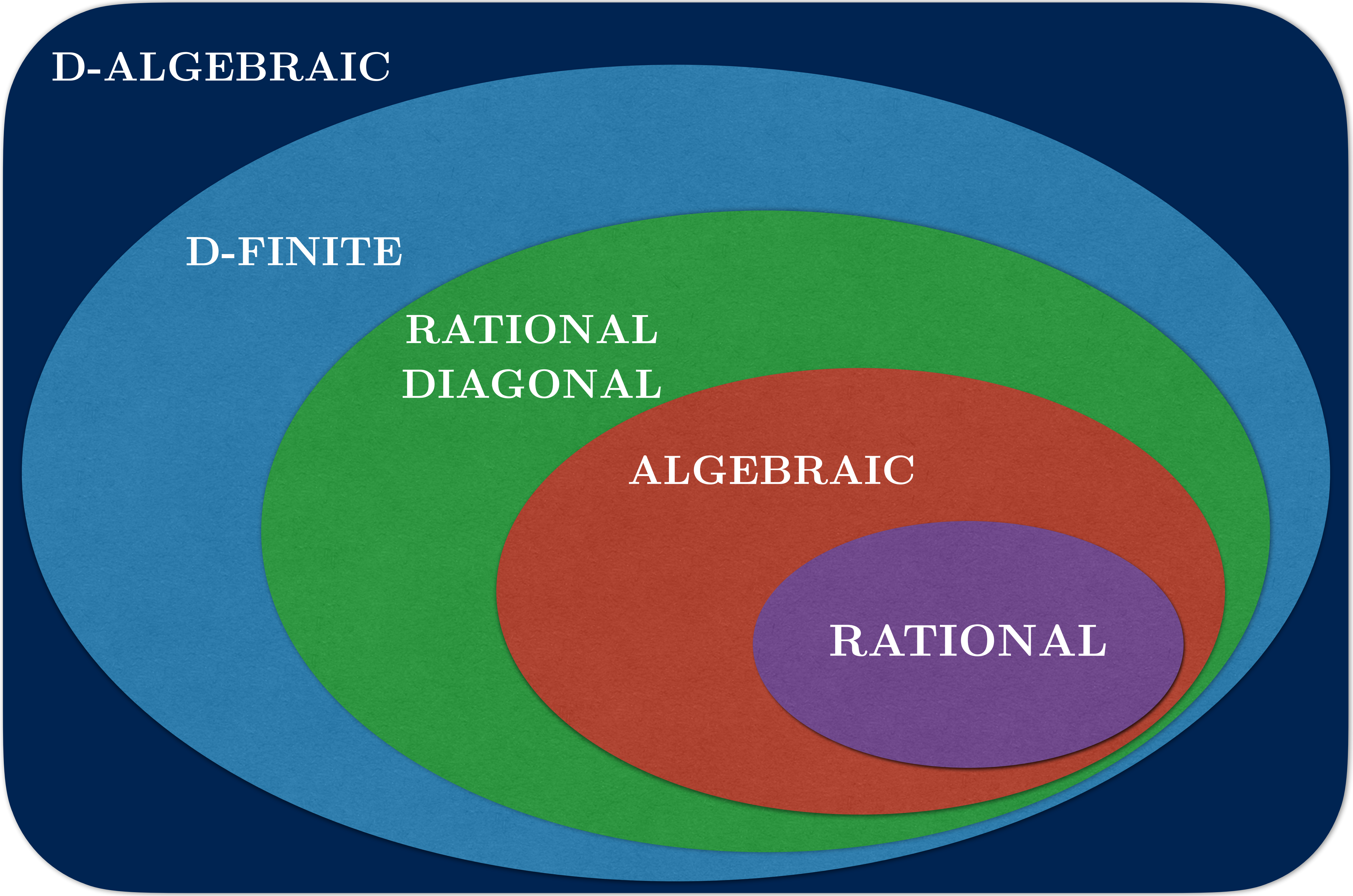
Why Diagonals?

- **Data structures** for interesting univariate sequences
- **Uniform asymptotics** over *most* directions (tomorrow)
- Yield **combinatorial limit theorems** (tomorrow)

We focus on **rational** (or **meromorphic**) diagonals

- Diagonal of an algebraic function in d variables is the diagonal of a rational function in *2d* variables

Generating Function Classes



Analytic Combinatorics in Several Variables

coprime ratio

Assume

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}}$$

converges in a neighbourhood of the origin.

The singularities of $F(\mathbf{z})$ are given by $\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}$.
Singularities **closest** to the origin are called **minimal points**.

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$\mathbf{w}$ minimal if and only if $H(\mathbf{w}) = 0$ and there is no $\mathbf{z}$ with

$$H(\mathbf{z}) = 0 \quad \text{and} \quad |z_j| < |w_j| \text{ for all } j$$

Analytic Combinatorics in Several Variables

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Multivariate Cauchy Integral Formula

$$f_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}} \qquad n\mathbf{r} \in \mathbb{N}^d$$

where $\mathcal{C}$ is a product of circles $|z_i| = \varepsilon$

Difficulties of ACSV

One variable rational (or meromorphic) functions

- Find finite set of singularities closest to the origin
- Add their asymptotic contributions

In more than one variable

- Set of minimal points is infinite
- Singular set can have nontrivial geometry (self-intersections)
- Can deform domain of integration *around* singular set!

Smooth ACSV

Simplest case: Denominator H and its partial derivatives don't simultaneously vanish.

Then **critical points** are defined by

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

partial derivative

Smooth ACSV

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Critical points: Asymptotic approximations can be made
Minimal points: Cauchy integral can be deformed close to

The asymptotic contribution of a minimal critical point $\mathbf{w}$ depends on an explicit matrix $\mathcal{M} = \mathcal{M}_{\mathbf{w}}$ built from partial derivatives of H

Surgery ACSV Theorem (Pemantle Wilson 2003)

Suppose that

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

admits a minimal solution $\mathbf{w} \in \mathbb{C}_*^d$. If

- no other singularity has the same coordinate-wise modulus as $\mathbf{w}$
- $H_{z_d}(\mathbf{w})$ and $\det \mathcal{M}$ are non-zero,

then

$$[\mathbf{z}^{n\mathbf{r}}]\frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}} \; (nr_d)^{(1-d)/2} \; (2\pi)^{(1-d)/2} \det(\mathcal{M})^{-1/2} \left(\frac{-G(\mathbf{w})}{w_d \, H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right)\right)$$

If there are a finite number of singularities with the same coordinate-wise modulus as $\mathbf{w}$, all satisfying these conditions, then we can add their asymptotic contributions.

The Hessian Matrix

If $H_{z_d}(\mathbf{w}) \neq 0$ then we can write $z_d = g(\widehat{\mathbf{z}})$ near $\mathbf{w}$

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$\mathcal{M}$ is the $(d-1) \times (d-1)$ Hessian matrix at $\theta = \mathbf{0}$ of

$$\phi(\theta) = \log g\left(w_1 e^{i\theta_1}, \ldots, w_{d-1} e^{i\theta_{d-1}}\right)$$

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The Chain Rule implies

$$\mathcal{M}_{i,j} = \begin{cases} V_i V_j + U_{i,j} - V_j U_{i,d} - V_i U_{j,d} + V_i V_j U_{d,d} & : i \neq j \\ V_i + V_i^2 + U_{i,i} - 2V_i U_{i,d} + V_i^2 U_{d,d} & : i = j \end{cases}$$

where $U_{i,j} = \dfrac{w_i w_j H_{z_i z_j}(\mathbf{w})}{w_d H_{z_d}(\mathbf{w})}$ and $V_i = r_i / r_d$

The Hessian Matrix

If $H_{z_d}(\mathbf{w}) \neq 0$ then we can write $z_d = g(\widehat{\mathbf{z}})$ near $\mathbf{w}$

$\mathcal{M}$ is the $(d-1) \times (d-1)$ Hessian matrix at $\theta = \mathbf{0}$ of

$$\phi(\theta) = \log g\left(w_1 e^{i\theta_1}, \ldots, w_{d-1} e^{i\theta_{d-1}}\right)$$

```
def getHes(H,R,vars,CP):
    dd = len(vars)
    V = zero_vector(SR,dd)
    U = matrix(SR,dd)
    M = matrix(SR,dd-1)

    for j in range(dd):
        V[j] = R[j]/R[-1]
        for i in range(dd):
            U[i,j] = vars[i]*vars[j]*diff(H,vars[i],vars[j])/vars[-1]/diff(H,vars[-1])
    for i in range(dd-1):
        for j in range(dd-1):
            M[i,j] = V[i]*V[j] + U[i,j] - V[j]*U[i,-1] - V[i]*U[j,-1] + V[i]*V[j]*U[-1,-1]
            if i == j: M[i,j] = M[i,j] + V[i]
    return M.subs(CP)
```

A First Example

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

Critical Point Equations in Direction $\mathbf{r} = (1,1)$

$$1 - x - y = 0 \qquad\qquad -x = -y$$

Unique Minimal Critical Point

$$(x_*, y_*) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Hessian

$$g(x) = 1 - x \ \text{ so } \ \phi(\theta) = \log\left(1 - \tfrac{1}{2}e^{i\theta}\right) \ \text{ and } \ \mathcal{M} = \phi''(0) = 2$$

A First Example

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

Critical Point Equations in Direction $\mathbf{r} = (1,1)$

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Asymptotics

$$\binom{2n}{n} = [x^n y^n] F(x,y) = \frac{4^n}{\sqrt{\pi n}} \left(1 + O\left(n^{-1}\right)\right)$$

A First Example

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{i} x^i y^j$$

Critical Point Equations in Direction $\mathbf{r} = (r,s)$

$$1 - x - y = 0 \qquad\qquad -sx = -ry$$

Unique Minimal Critical Point

$$(x_*, y_*) = \left(\frac{r}{r+s}, \frac{s}{r+s}\right)$$

Asymptotics

$$\binom{rn+sn}{rn} = [x^{rn}y^{sn}]F(x,y) = \left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}\left(1+O\left(n^{-1}\right)\right)$$

Bi-Clover Quiver
(Ramgoolam, Wilson and Zahabi 2020)

The generating function for the *chiral operators in the large N limit of the bi-clover quiver gauge theory* is

$$F(x,y) = \frac{1}{\prod_{k\geq 1}(1 - x^k - y^k)}$$

Bi-Clover Quiver (Ramgoolam, Wilson and Zahabi 2020)

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$$F(x,y) = \frac{1}{\prod_{k\geq 1}(1-x^k-y^k)}$$

Note: $F(x,y) = G(x,y)/(1-x-y)$ where

$$G(x,y) = \prod_{k\geq 2}(1-x^k-y^k)^{-1}$$

Thus,

$$[x^{rn}y^{sn}]F(x,y) = G\left(\frac{r}{r+s}, \frac{s}{r+s}\right)\left(\frac{r+s}{r}\right)^{rn}\left(\frac{r+s}{s}\right)^{sn}\frac{\sqrt{r+s}}{\sqrt{2rs\pi\, n}}\left(1+O\left(n^{-1}\right)\right)$$

Proof Idea (Bivariate)

Suppose $\mathbf{w} = (a, b) \in \mathbb{R}^2_{>0}$ satisfies the conditions of the theorem.

Cauchy Integral Formula implies

$$f_{rn,sn} = \frac{1}{(2\pi i)^2} \int_{|x|=a} \left(\int_{|y|=b-\varepsilon} F(x,y) \frac{dx\,dy}{x^{rn+1} y^{sn+1}} \right)$$

Proof Idea (Bivariate)

Suppose $\mathbf{w} = (a, b) \in \mathbb{R}^2_{>0}$ satisfies the conditions of the theorem.

Cauchy Integral Formula and **Max Modulus Bound** imply

$$f_{rn,sn} = \frac{1}{(2\pi i)^2} \int_{|x|=a} \left(\int_{|y|=b-\varepsilon} F(x,y) \frac{dx\, dy}{x^{rn+1} y^{sn+1}} \right)$$
$$- \frac{1}{(2\pi i)^2} \int_{|x|=a} \left(\int_{|y|=b+\varepsilon} F(x,y) \frac{dx\, dy}{x^{rn+1} y^{sn+1}} \right) + \text{small error}$$

* *May need to localize*

Proof Idea (Bivariate)

Suppose $\mathbf{w} = (a, b) \in \mathbb{R}^2_{>0}$ satisfies the conditions of the theorem.

Cauchy Integral Formula and **Max Modulus Bound** and **Residues** imply

$$f_{rn,sn} = \frac{1}{2\pi i} \int_{|x|=a} \left(\operatorname*{Res}_{y=g(x)} \frac{F(x,y)}{y^{sn+1}} \right) \frac{dx}{x^{rn+1}} + \text{small error}$$

where $y = g(x)$ on $\mathcal{V}$

* *May need to localize*

Proof Idea (Bivariate)

Suppose $\mathbf{w} = (a, b) \in \mathbb{R}^2_{>0}$ satisfies the conditions of the theorem.

Cauchy Integral Formula and **Max Modulus Bound** and **Residues** imply

$$f_{rn,sn} = \frac{1}{2\pi i} \int_{|x|=a} \frac{G(x, g(x))}{H_y(x, g(x))} \cdot \frac{dx}{x^{rn+1} g(x)^{sn+1}} + \text{small error}$$

where $y = g(x)$ on $\mathcal{V}$

* *May need to localize*

Proof Idea (Bivariate)

Suppose $\mathbf{w} = (a, b) \in \mathbb{R}^2_{>0}$ satisfies the conditions of the theorem.

Cauchy Integral Formula and **Max Modulus Bound** and **Residues** and the **change of variables** $x = ae^{i\theta}$ imply

$$f_{rn,sn} = \int_{-\pi}^{\pi} A(\theta) e^{-n\phi(\theta)} d\theta + \text{small error}$$

Because $\mathbf{w}$ is **critical** this is a **saddle-point integral**

* *May need to localize*

Critical vs Minimal Points

Minimal points make the proof relative **easy**, but make checking the conditions **difficult**.

Problem: Dealing with minimality (and points with same coordinate-wise modulus) is hard. It also considers spurious points.

$$F(x,y) = \frac{1}{(1+2y)(1-x-y)}$$ still has critical point $(1/2, 1/2)$

But now there is a **curve** of singularities

$$\left\{ \left(e^{i\theta}/2, -1/2 \right) : \theta \in (-\pi, \pi] \right\}$$

with $|x| = |y| = 1/2$

Critical vs Minimal Points

Minimal points make the proof relative **easy**, but make checking the conditions **difficult**.

Problem: Dealing with minimality (and points with same coordinate-wise modulus) is hard. It also considers spurious points.

Solution: Reduce importance of minimality. Only *critical points* really matter when computing asymptotics.

Key: *Generically* there are a finite set of critical points, encoded by algebraic equations, even though there are infinite minimal points.

Main Theorem of Smooth ACSV

(Baryshnikov Pemantle 2011 / BMP 2021)

Suppose that

$$H = 0, \qquad r_j z_1 H_{z_1} = r_1 z_j H_{z_j} \quad (2 \leq j \leq d)$$

admits a finite number of solutions. If

- there is exactly one minimal solution, $\mathbf{w} \in \mathbb{C}_*^d$
- $H_{z_d}(\mathbf{w})$ and $\det \mathcal{M}$ are non-zero,

then

$$[\mathbf{z}^{n\mathbf{r}}]\frac{G(\mathbf{z})}{H(\mathbf{z})} = \mathbf{w}^{-n\mathbf{r}}\,(nr_d)^{(1-d)/2}\,(2\pi)^{(1-d)/2}\det(\mathcal{M})^{-1/2}\left(\frac{-G(\mathbf{w})}{w_d\,H_{z_d}(\mathbf{w})} + O\left(\frac{1}{n}\right)\right)$$

In other words: We can study the (hopefully finite) set of critical points and check which are minimal, ignoring everything else

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$$[x^n y^n]\frac{1}{(1+2y)(1-x-y)} \sim \frac{1}{2} \cdot \frac{4^n}{\sqrt{\pi n}}$$

Main Theorem of Smooth ACSV

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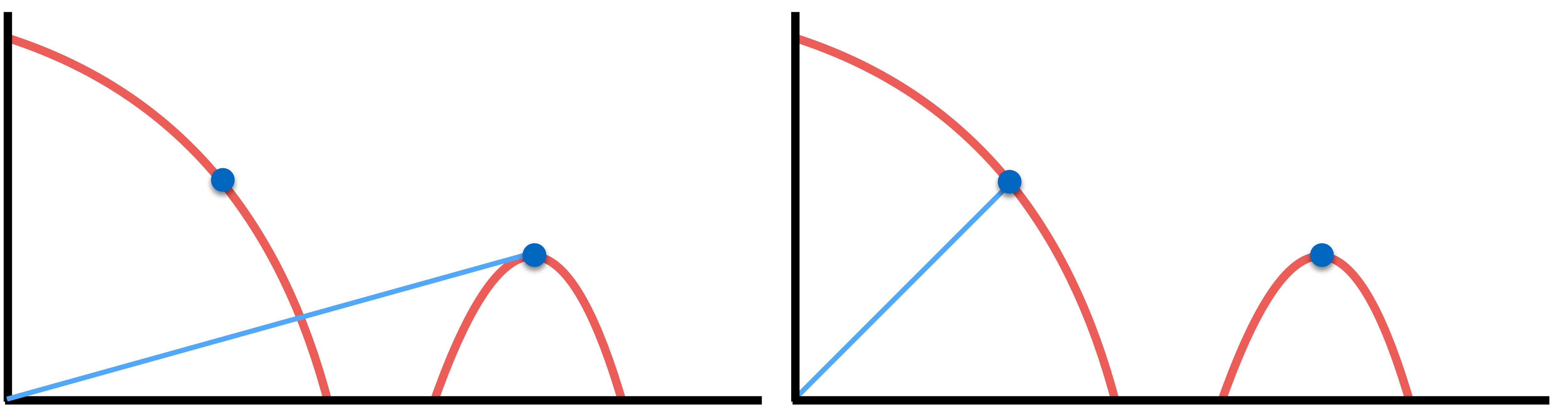
If there are a finite number of **critical points** with the same coordinate-wise modulus as $\mathbf{w}$, all satisfying these conditions, then we can add their asymptotic contributions.

Help Proving Minimality

Multivariate Vivanti-Pringsheim Theorem
If $f_{\mathbf{i}} \geq 0$ for all $\mathbf{i}$ then $\mathbf{w} \in \mathcal{V}$ with positive coordinates is minimal if and only if $H(t\mathbf{w}) \neq 0$ for all $t \in (0,1)$.



Help Proving Minimality

Aperiodic Expansions

If $H(\mathbf{z}) = 1 - \sum_{\mathbf{n}\in\mathbb{N}^d} p_{\mathbf{n}}\mathbf{z}^{\mathbf{n}}$ where $(p_{\mathbf{n}})$ is a sequence of nonnegative numbers with $\mathrm{span}_{\mathbb{Z}}\{\mathbf{n} \in \mathbb{N}^d : p_{\mathbf{n}} \neq 0\} = \mathbb{Z}^d$ then every minimal point has positive real coordinates

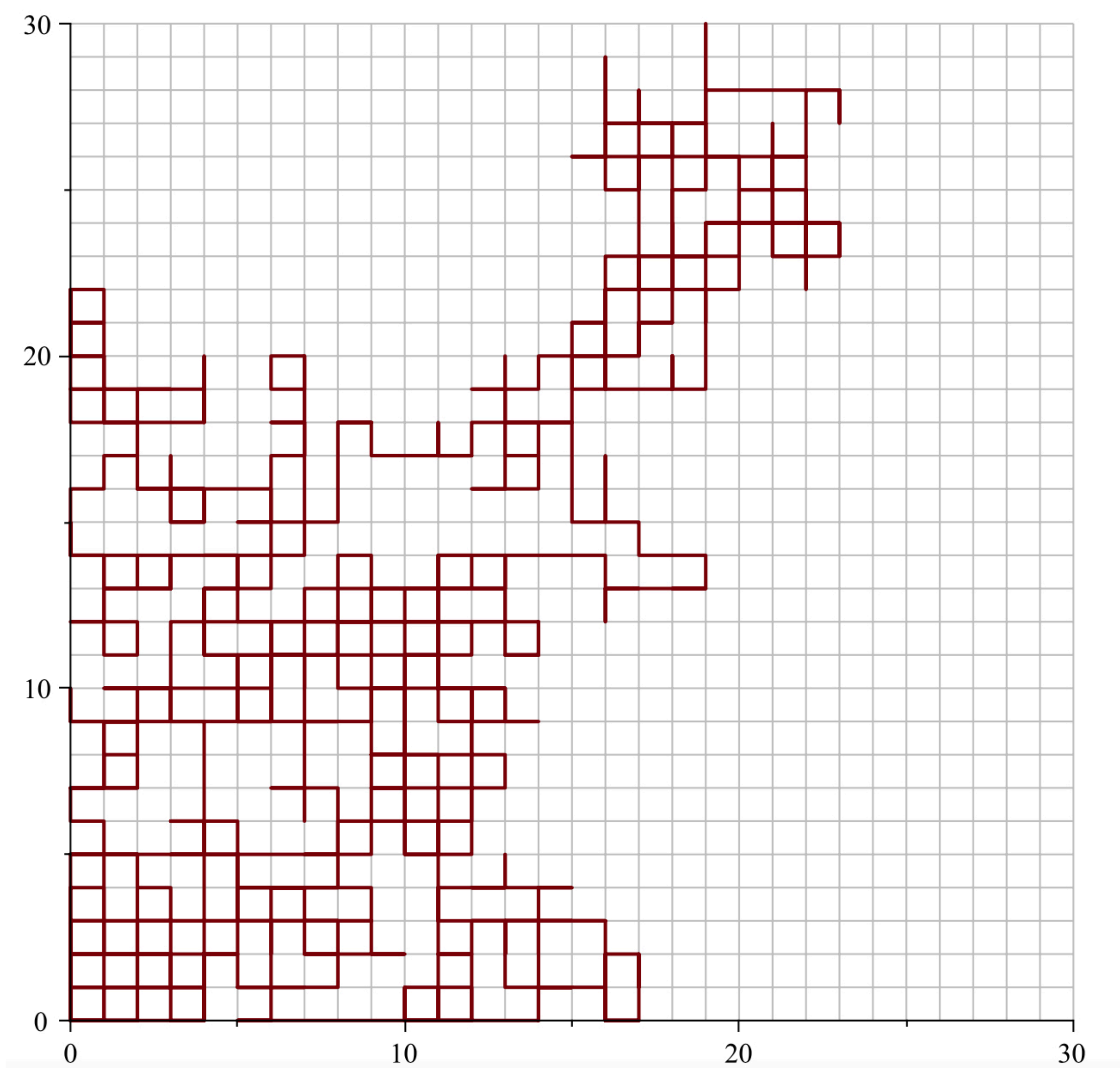
$$\frac{1}{1-x-y} \qquad \frac{1}{2-e^{x+y}}$$

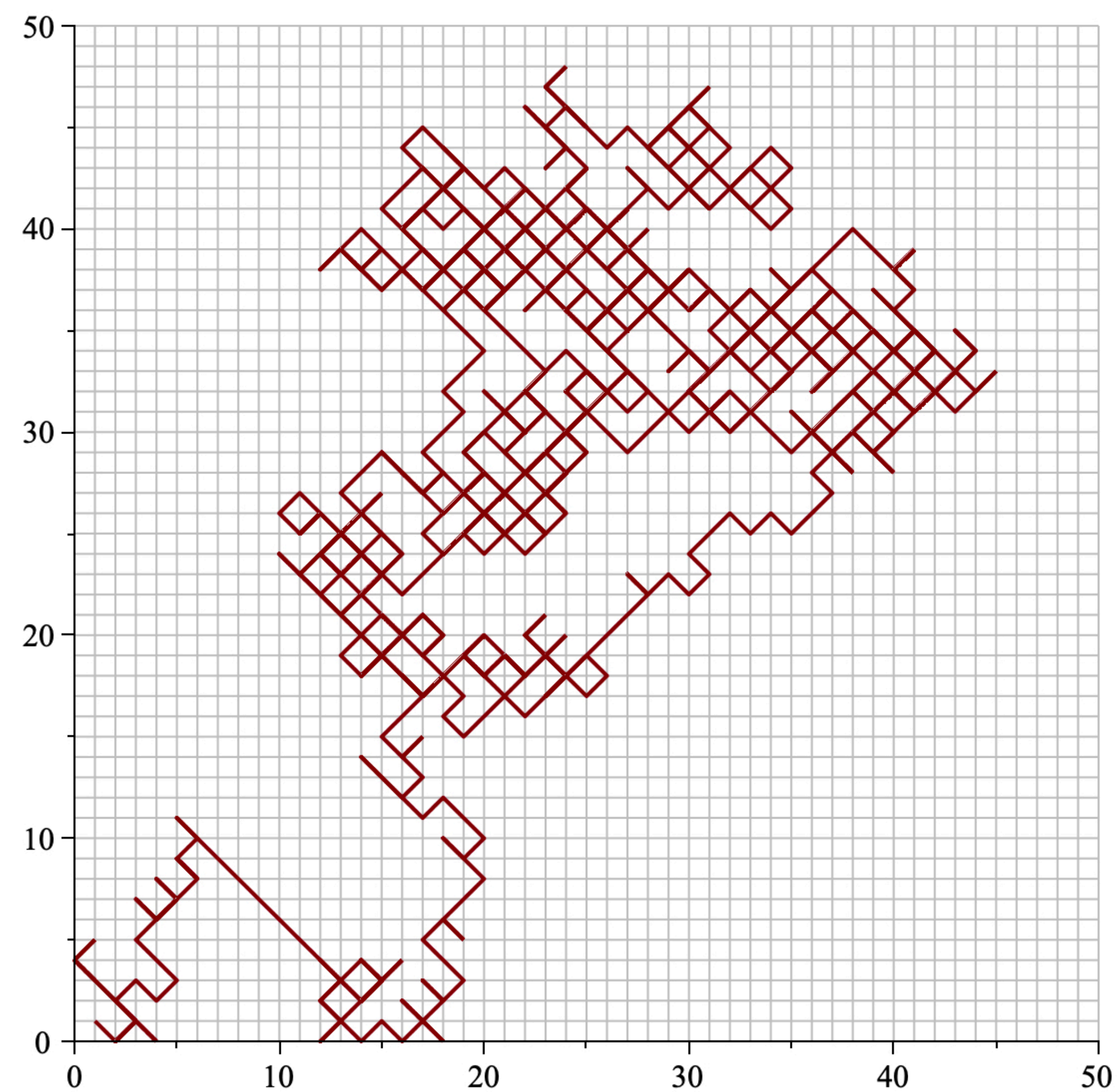
$$\frac{1}{1-t(x+y)} \qquad \frac{1}{1-x+y}$$

Walks in an Orthant

Uniform diagonal expression for walk models in $\mathbb{N}^d$ whose **step sets** $\mathcal{S} \subset \{\pm 1, 0\}^d$ **are symmetric over every axis**.

$$[(z_1 \cdots z_d t)^n] \, \frac{(1+z_1)\cdots(1+z_d)}{1 - t(z_1 \cdots z_d) S(\mathbf{z})}, \qquad S(\mathbf{z}) = \sum_{\mathbf{i} \in \mathcal{S}} \mathbf{z}^{\mathbf{i}}$$



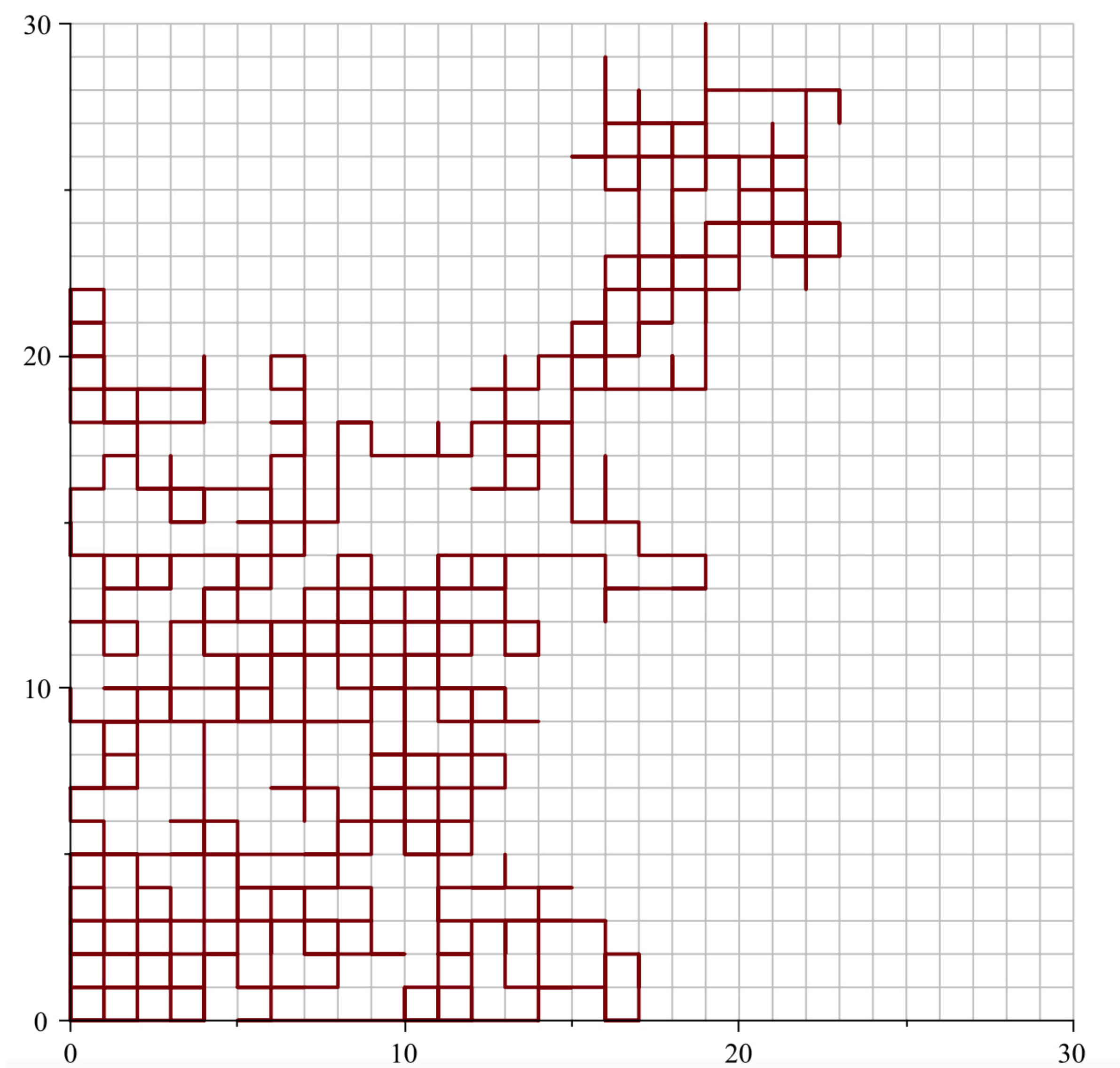


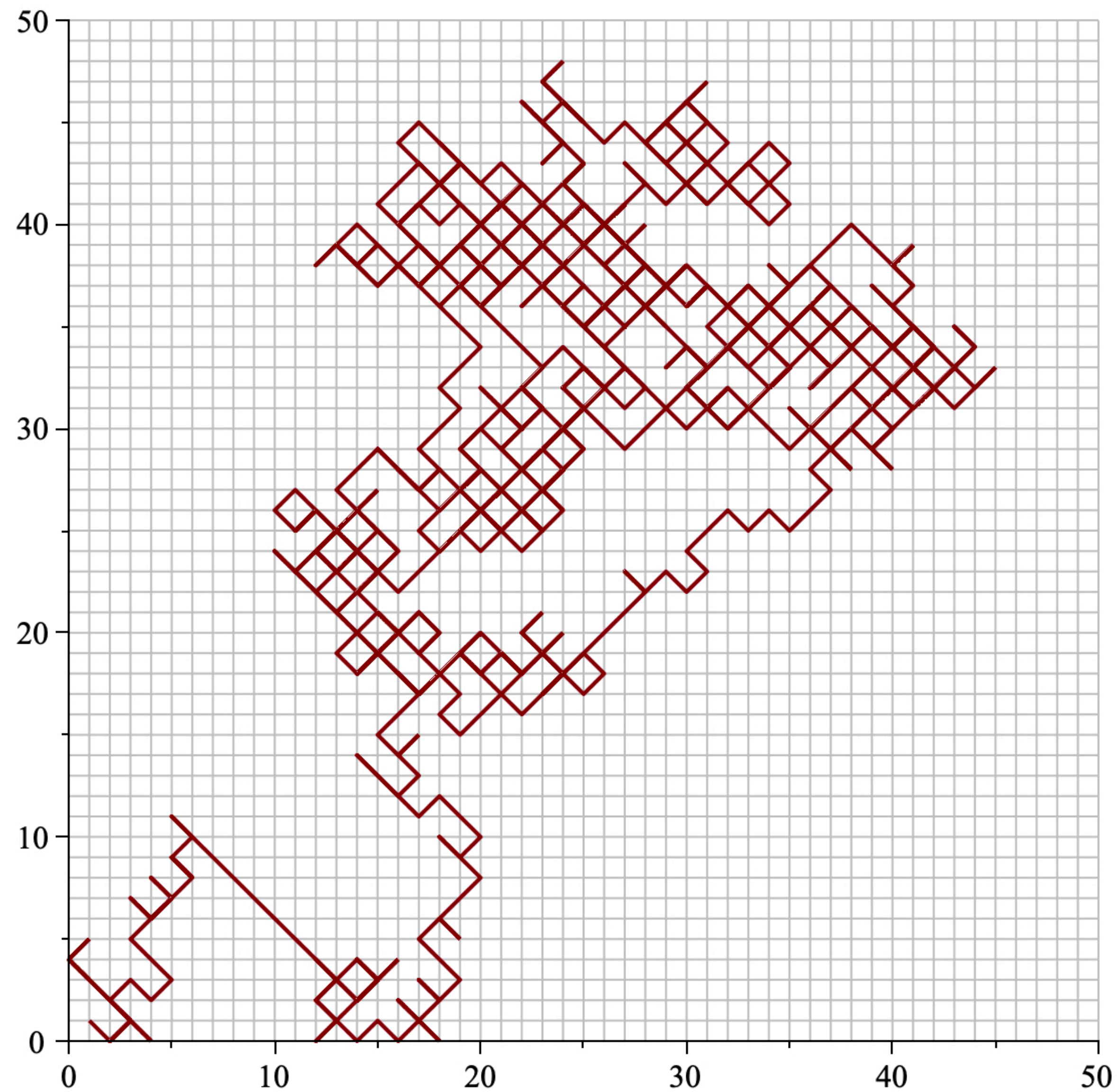
Walks in an Orthant

Uniform diagonal expression for walk models in $\mathbb{N}^d$ whose **step sets** $\mathcal{S} \subset \{\pm 1, 0\}^d$ **are symmetric over every axis**.

$$\# \text{ walks} \sim |\mathcal{S}|^n \cdot n^{-d/2} \cdot \left(\left(s^{(1)} \cdots s^{(d)} \right)^{-1/2} \pi^{-d/2} |\mathcal{S}|^{d/2} + O\left(\frac{1}{n}\right) \right)$$

M. and Mishna, 2016





Lonesum Matrices (Khera, Lundberg, and M.)

A **lonesum matrix** is a $0-1$ matrix that is uniquely determined by its row and column sums.

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

NO

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}$$

$$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$$

YES

Lonesum Matrices (Khera, Lundberg, and M.)

A **lonesum matrix** is a $0-1$ matrix that is uniquely determined by its row and column sums.

$$F(x,y) = \sum_{n,k\geq 0} \frac{B_{n,k}}{n!k!} x^n y^k = \frac{1}{e^{-x} + e^{-y} - 1}$$

Noncommutative Biology: Sequential Regulation of Complex Networks
Letsou and Cai. PLOS Computational Biology, 2016.

Together with the fact that the reachable configurations are a subset of the staircase matrices, this implies that the reachable configurations and the lonesum matrices are in fact the same set, and we have

Theorem 3 *The number of reachable configurations in the* (n, m) *ratchet network with* $l_n = l_m = 1$ *and threshold 1 scales as the poly-Bernoulli numbers* $B_m^{-n} = B_n^{-m}$.

Lonesum Matrices (Khera, Lundberg, and M.)

A **lonesum matrix** is a $0-1$ matrix that is uniquely determined by its row and column sums.

$$F(x,y)=\sum_{n,k\geq 0}\frac{B_{n,k}}{n!k!}x^n y^k=\frac{1}{e^{-x}+e^{-y}-1}$$

Let $f(t)=t/(1-e^t)\log(1-e^{-t})$.

Theorem. If $n,k\to\infty$ such that $n/k\to\lambda>0$ then

$$B_{n,k}=\frac{a^{-n}b^{-n}}{\sqrt{k}}\frac{n!k!}{\sqrt{2\pi a e^{-a}[be^{-b}+ae^{-a}-ab]}}\left(1+O\left(k^{-1}\right)\right),$$

where $a=f^{-1}(\lambda)$ and $b=f^{-1}(1/\lambda)$

Proof Idea of Main Theorem

Approach 1: Cones of Hyperbolicity (BP 2011)

- Criticality says something about the tangent space to $\mathcal{V}$
- Use this to locally deform around non-critical minimal points
- Glue these deformations together with roots of unity

Approach 2: Intersection Classes (BMP 2022)

- Use geometric arguments to "flow" cycle on $\mathcal{V}$
- More on this approach tomorrow

ACSV Complexity Results

Suppose that $G(\mathbf{z})$ and $H(\mathbf{z})$ have coefficients $\leq 2^h$ and degree q

Suppose also that the power series of $F(\mathbf{z})$ has non-negative coefficients

Theorem (M. and Salvy, 2016)
Under generic and verifiable assumptions one can **find all minimal critical points**, and compute asymptotics in $\tilde{O}(hq^{4d+5})$ bit operations.

Can remove non-negativity assumption, with increased complexity.

Theorem (M. and Salvy, 2021)
Under verifiable assumptions, one can find minimal critical points in $\tilde{O}(hq^{9d+4}2^{3d})$ bit operations.

ACSV Complexity Results

Theorem (M. and Salvy, 2021)
Under verifiable assumptions, one can find minimal critical points in $\tilde{O}(hq^{9d+4}2^{3d})$ bit operations.

$O(hd^{9n+4}2^{3n})$

General Idea:

- Assumptions imply finite number of critical points

- Use a *univariate (Kronecker) representation* to encode them

- Reduce everything to **polynomial equalities and inequalities with bounded degrees and coefficient sizes**

- Use numerical methods with sufficient precision to test minimality

Irrationality of Zeta(3)

Exercise

Be the first in your block to prove by a 2-line argument that $\zeta(3)$ is irrational.[7]

(6) Given the definitions of (5) show that $a_n b_{n-1} - a_{n-1} b_n = b_n^{-3}$ and $b_n = O(\alpha^n)$ with $\alpha = (1+\sqrt{2})^4$. Conclude that $\zeta(3)$ is irrational because $\log \alpha > 3$.

A Proof that Euler Missed, Alfred van der Poorten

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = [(xyzt)^n] \frac{1}{1 - t(1+x)(1+y)(1+z)(1+y+z+yz+xyz)}$$

Irrationality of Zeta(3)

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A Proof that Euler Missed, Alfred van der Poorten

```
> A, U, PRINT := DiagonalAsymptotics(numer(F),denom(F),[a,b,c,z],u,k, useFGb):
  A, U;
```

$$\frac{1}{4}\,\frac{\left(\frac{2u-366}{34u+1458}\right)^{k}\sqrt{2}\,\sqrt{\frac{2u-366}{-96u-4192}}}{k^{3/2}\,\pi^{3/2}},\ \left[RootOf\left(\_Z^2 - 366\,\_Z - 17711,\ -43.27416997969\right.\right.$$

Restricted Factors in Words

The number of *balanced* binary strings with no substring equal to 10101101 and 1110101 is the main diagonal of

$$\frac{1 - x^3y^6 + x^3y^4 + x^2y^4 + x^2y^3}{1 - x - y + x^2y^3 - x^3y^3 - x^4y^4 - x^3y^6 + x^4y^6}$$

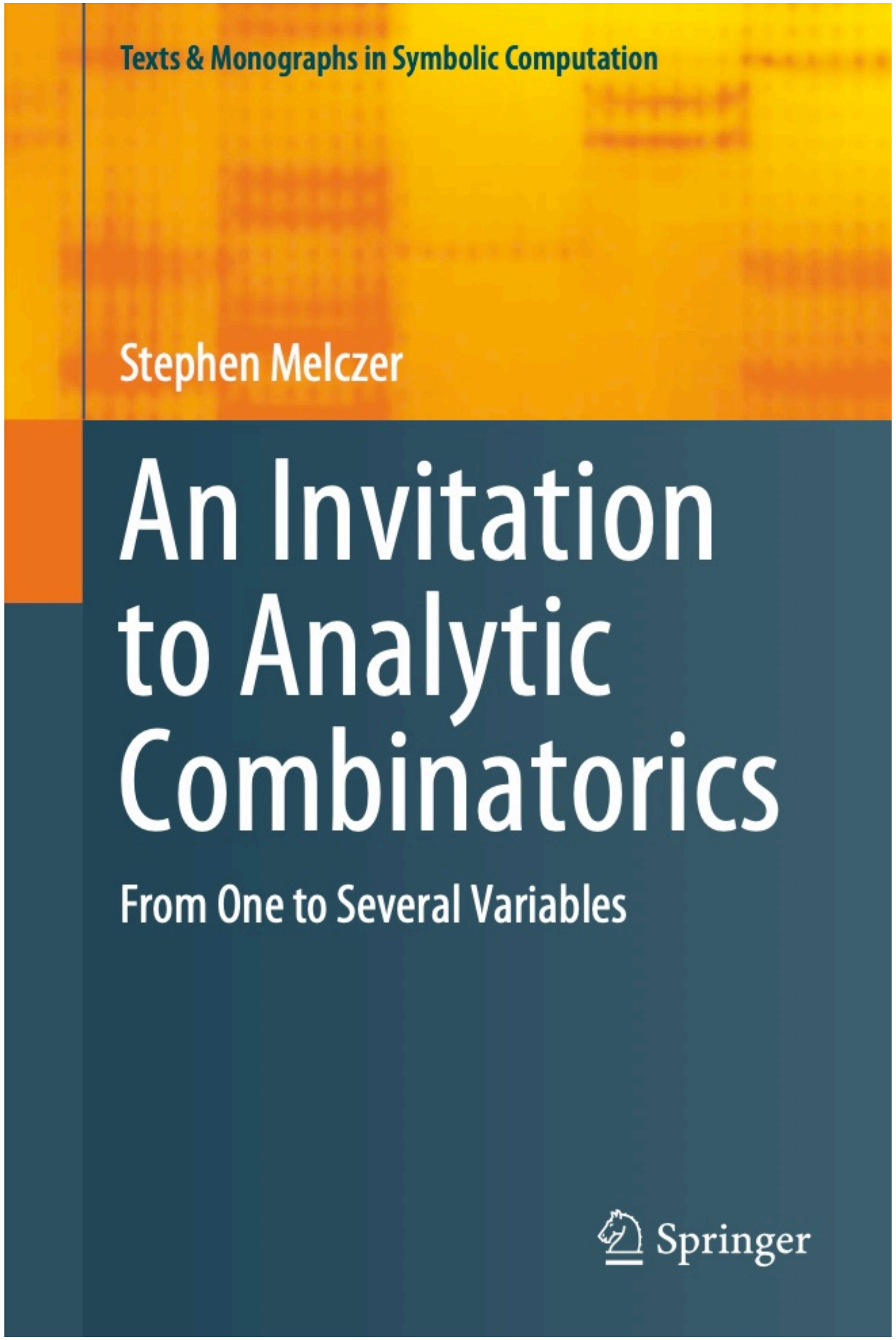
```
> ASM, U := DiagonalAsymptotics(numer(F),denom(F),indets(F),u,k,true,u-T,T):
  ASM;
```

$$\frac{1}{2}\Bigg(\Bigg(\frac{84 u^{20} + 240 u^{19} - 285 u^{18} - 1548 u^{17} - 2125 u^{16} - 1408 u^{15} + 255 u^{14} + 756 u^{13} + 2509 u^{12} + 2856 u^{11} + 605 u^{10} + 2020 u^{9} + 1233 u^{8} - 1760 u^{7} +}{-12 u^{20} + 30 u^{19} + 258 u^{18} + 500 u^{17} + 440 u^{16} - 102 u^{15} - 378 u^{14} - 1544 u^{13} - 2142 u^{12} - 550 u^{11} - 2222 u^{10} - 1644 u^{9} + 2860 u^{8} - 1848 u^{7} + 123}$$

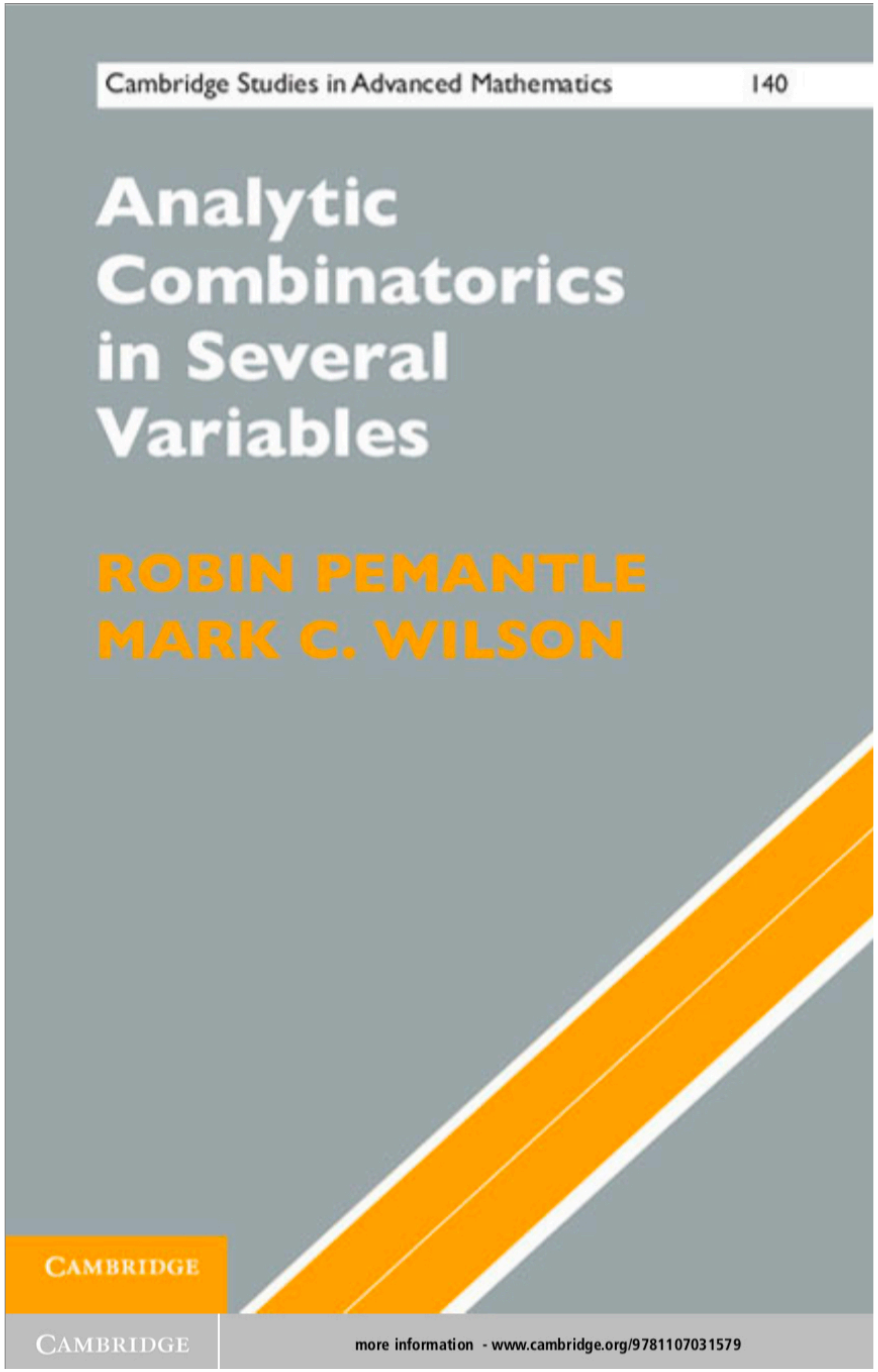
$$\sqrt{\frac{84 u^{20} + 240 u^{19} - 285 u^{18} - 1548 u^{17} - 2125 u^{16} - 1408 u^{15} + 255 u^{14} + 756 u^{13} + 2509 u^{12} + 2856 u^{11} + 605 u^{10} + 2020 u^{9} + 1233 u^{8} - 1760 u^{7} + 9}{-162 u^{18} - 612 u^{17} - 902 u^{16} - 616 u^{15} + 254 u^{14} + 548 u^{13} + 2054 u^{12} + 2156 u^{11} + 898 u^{10} + 2268 u^{9} + 2462 u^{8} - 2088 u^{7} + 1312 u^{6} -}}$$

$$- 255 u^{16} - 190 u^{15} - 19 u^{14} + 46 u^{13} + 461 u^{12} + 628 u^{11} + 133 u^{10} + 374 u^{9} + 161 u^{8} - 384 u^{7} + 146 u^{6} - 138 u^{5} - 285 u^{4} - 40 u^{3} + 91 u^{2} - 30 u + 32$$

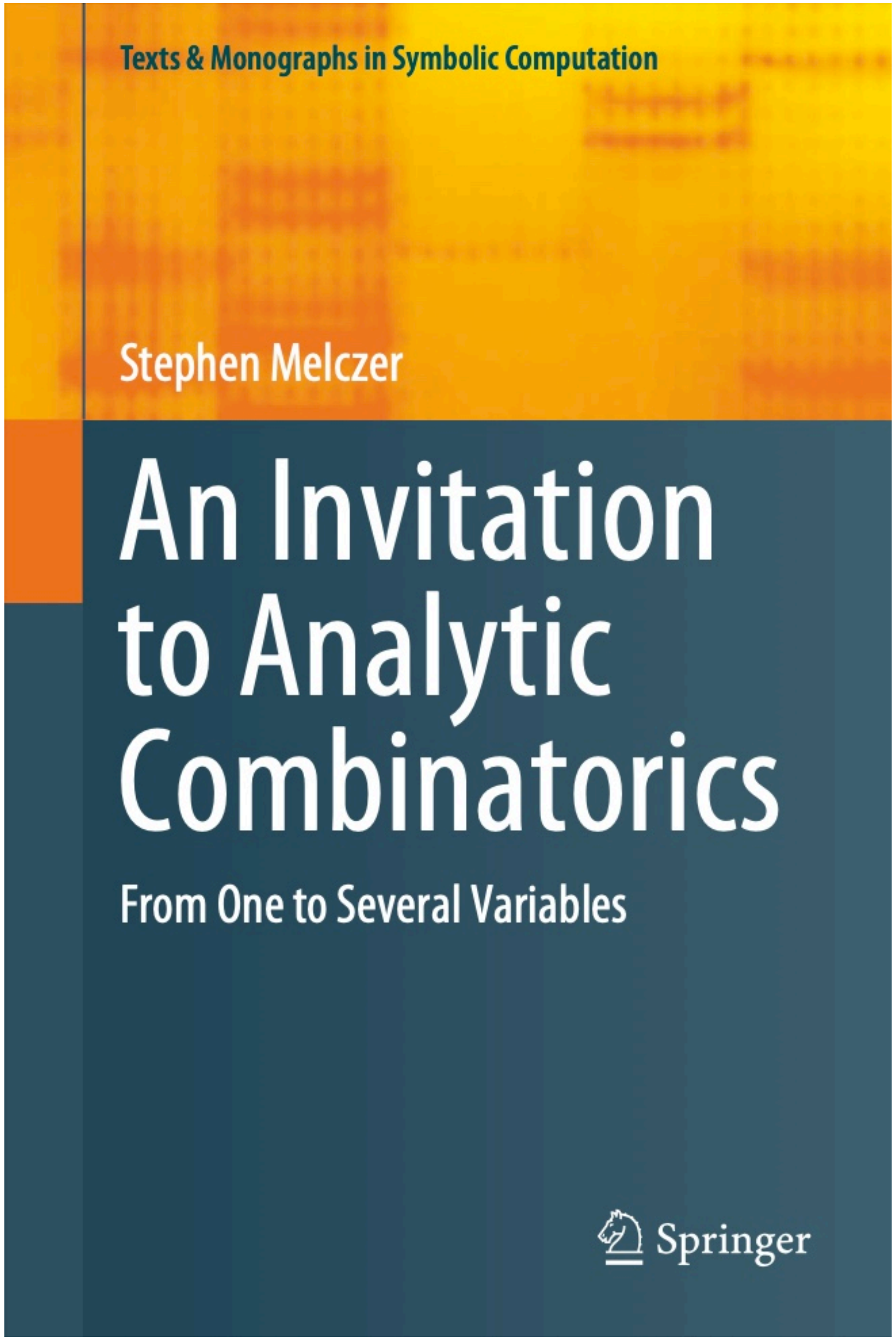
$$+ 756 u^{13} + 2509 u^{12} + 2856 u^{11} + 605 u^{10} + 2020 u^{9} + 1233 u^{8} - 1760 u^{7} + 924 u^{6} - 492 u^{5} - 675 u^{4} + 632 u^{3} - 249 u^{2} + 24 u + 16\Bigg)\Bigg)$$

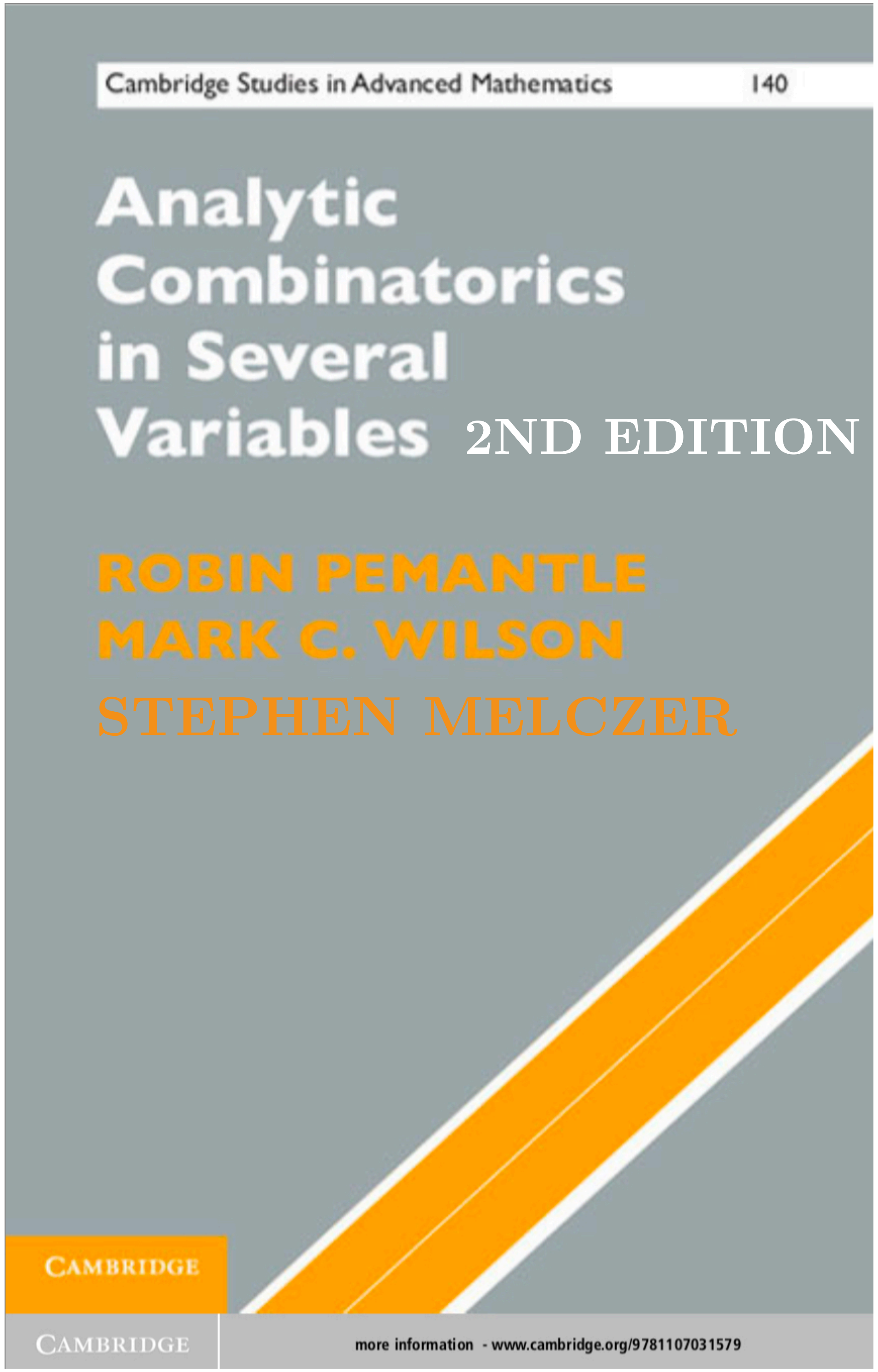


For new researchers — focus more on explicit results and computation



Most general theory, centred around topological framework





For new researchers — focus more on explicit results and computation

Most general theory, centred around topological framework

Implementations

- **Sage package** of Alex Raichev for computing asymptotic contributions of (already certified) minimal critical points

  - Package was not well maintained. Currently being fixed and extended to certify minimality by Hackl, Selover, and Wong.

- **Julia implementation** using Homotopy Continuation to certify minimality by Kisun Lee and Josip Smolić.

- **Maple implementation** of Melczer and Salvy for certifying minimality.

- **Sage code** developed for *An Invitation to Analytic Combinatorics* (available on melczer.ca/textbook)

Conclusion

Conclusion

- Analytic combinatorics is beautiful and powerful
- Analytic combinatorics in several variables is beautiful and powerful
- You don’t need much more than univariate analytic combinatorics to get interesting results
- To remove some hard-to-check hypotheses we need to bring in new techniques and rely on some advanced mathematics

Lecture 2 (Tomorrow)

- Uniform asymptotics in varying directions
- Limit theorems
- (Some) non-smooth singular sets
- Morse-theoretic framework
- New applications
- And more!

THANK YOU!

An Invitation to Analytic Combinatorics
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